CKMfitter and CKMlive

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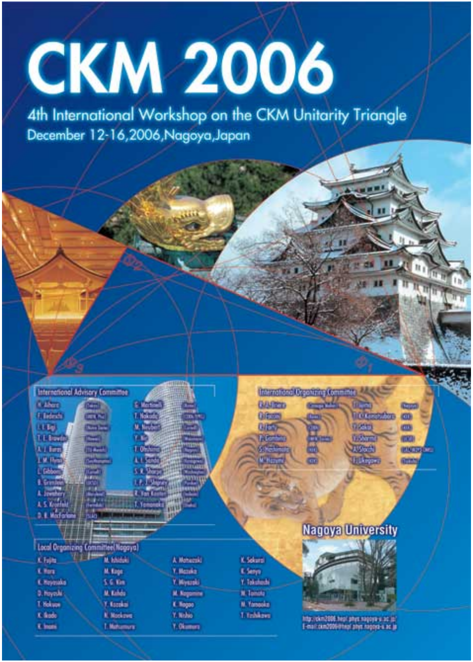

CKM 2006
4th International Workshop on the CKM Unitarity Triangle
December 12-16,2006,Nagoya,Japan
International Advisory Committee
International Organizing Committee
Local Organizing Committee(Nagoya)
Nagoya University

Global format

Objectives

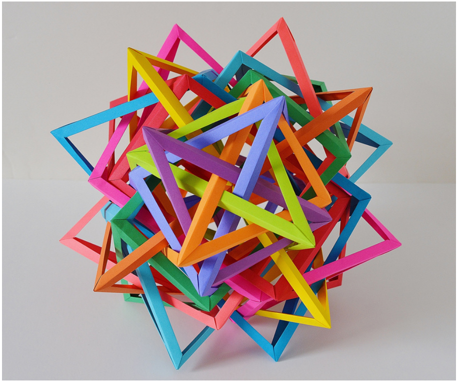
- Complement lectures on CP-violation with practical sessions
- Understand how we know the amount of CP-violation in SM
- Illustrate the challenges of extracting theoretical info from pheno

Outline of the three sessions

- Determining the CKM matrix parameters (physics and statistics)
- Implementing the approach in software (CKMfitter and 1st tutorial)
- Using the web-based interface (CKMlive (2nd tutorial)

Please get Firefox and go to `http://ckmlive.in2p3.fr`
in order to register (sign in) and be ready for tomorrow's session

CKM, or a story of triangles



with my apologies to Yossi and to all of you for the repetitions

Standard Model and weak interaction

SM: $SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$

- Colour (for quarks only)
- Weak isospin (for left-handed fermions only)
- Hypercharge (for everybody)

- Interactions in covariant derivatives of kinetic terms, written in terms of three distinct generations of interaction eigenstates

$$\mathcal{L} = i \sum_J \bar{\psi}_J \not{D} \psi_J + \ldots \qquad \psi_J = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, u_R, d_R, c_R \ldots$$

- After electroweak symmetry breaking, mass eigenstates ψ', not necessarily identical to interaction eigenstates ψ:

$$u_L = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L = V_u \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L \qquad d_L = \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = V_d \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L$$

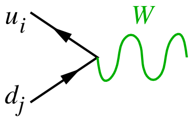
$\Longrightarrow$(Unitary) rotations may not align: $V_u \neq V_d$ (ditto for u_R, d_R)

FCCC: flavour-changing charged currents

- W bosons couple to charged currents J_W^μ
- which in mass eigenstate basis involve matrix V

$$J_W^\mu = \bar{u}_L^i \gamma^\mu d_L^i \to \bar{u}_L' V_u^\dagger \gamma^\mu V_d d_L' = \bar{u}_L' V \gamma^\mu d_L'$$

- flavour-changing charged currents at tree level



$$\frac{g}{\sqrt{2}} [\bar{u}_L^i V_{ij} \gamma^\mu d_L^j W_\mu^+ + \bar{d}_L^j V_{ij}^* \gamma^\mu u_L^i W_\mu^-]$$

unitary Cabibbo-Kobayashi-Maskawa matrix
(linked to electroweak symmetry breaking)

FCNC: flavour-changing neutral currents

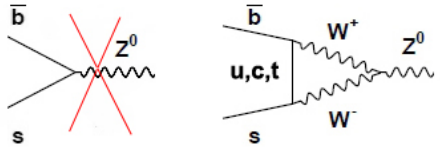
Neutral currents remain flavour-diagonal (same for u_R, d_R)

$$\sum_i \bar{u}_L^i \gamma^\mu u_L^i \to \sum_{ij} \bar{u}_L^{\prime j} V^\dagger_{u,ji} \gamma^\mu V_{u,ij} u_L^{\prime j} = \sum_j \bar{u}_L^{\prime j} \gamma^\mu u_L^{\prime j},$$

$$\sum_i \bar{d}_L^i \gamma^\mu d_L^i \to \sum_{ij} \bar{d}_L^{\prime j} V^\dagger_{d,ji} \gamma^\mu V_{d,ij} d_L^{\prime j} = \sum_j \bar{d}_L^{\prime j} \gamma^\mu d_L^{\prime j},$$

No flavour-changing neutral currents in SM

. . . but only at tree level ! They can occur in loops (but suppressed)



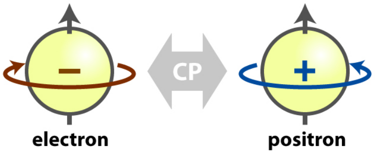
- Loop: Higher order in pert. theory (powers of g, g')
- GIM: Vanish in degenerate case $m_u = m_c = m_t$

  (proportional to $V^*_{tb} V_{ts} + V^*_{cb} V_{cs} + V^*_{ub} V_{us} = 0$)

CP and CKM

C (Charge conjugation) and P (Parity) combined in CP

- $\bar\psi_1\gamma_\mu(1-\gamma_5)\psi_2 \to \bar\psi_2\gamma_\mu(1-\gamma_5)\psi_1$
  $\bar\psi_1\gamma_\mu(1+\gamma_5)\psi_2 \to \bar\psi_2\gamma_\mu(1+\gamma_5)\psi_1$
  at $(\vec{x},t)$ at $(-\vec{x},t)$
- symmetry of QCD/QED, but
  symmetry for weak interactions ?



$$
\begin{aligned}
& W^+_\mu \bar u_i V_{ij}\gamma^\mu(1-\gamma_5)d_j + W^-_\mu \bar d_j V^*_{ij}\gamma^\mu(1-\gamma_5)u_i \\
\to CP \to \quad & W^-_\mu \bar d_i V_{ij}\gamma^\mu(1-\gamma_5)u_j + W^+_\mu \bar u_j V^*_{ij}\gamma^\mu(1-\gamma_5)d_i \\
= \quad & W^+_\mu \bar u_i V^*_{ij}\gamma^\mu(1-\gamma_5)d_j + W^-_\mu \bar d_j V_{ij}\gamma^\mu(1-\gamma_5)u_i
\end{aligned}
$$

Weak interactions are CP-invariant if V is real

Arbitrariness in field redefs means that for N_g generations, V contains

$\frac{(N_g-1)(N_g-2)}{2}$ phases and $\frac{N_g(N_g-1)}{2}$ moduli

CKM matrix and CP violation

For two generations, 1 modulus, no phase, no CP violation (Cabbibo)

$$V = \begin{bmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

For three generations, 3 moduli and 1 phase,
a unique source of CP violation in quark sector (Kobayashi-Maskawa)

$$V = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \simeq \begin{bmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\bar\rho - i\bar\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\bar\rho-i\bar\eta) & -A\lambda^2 & 1 \end{bmatrix} + O(\lambda^4)$$

Wolfenstein params exploiting observed hierarchy of matrix elements

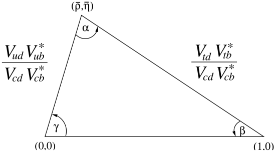
$\Longrightarrow$ extremely predictive model for CP violation embedded in SM

SM unitarity triangles

Many unitarity relations, e.g., related to 4 neutral mesons (no top)

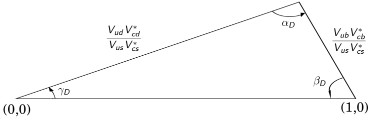
- B_d meson (bd) : $V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0$ $(\lambda^3, \lambda^3, \lambda^3)$
- B_s meson (bs) : $V_{us}V^*_{ub} + V_{cs}V^*_{cb} + V_{ts}V^*_{tb} = 0$ $(\lambda^4, \lambda^2, \lambda^2)$
- K meson (sd) : $V_{ud}V^*_{us} + V_{cd}V^*_{cs} + V_{td}V^*_{ts} = 0$ $(\lambda, \lambda, \lambda^5)$
- D meson (cu) : $V_{ud}V^*_{cd} + V_{us}V^*_{cs} + V_{ub}V^*_{cb} = 0$ $(\lambda, \lambda, \lambda^5)$

Representation of CKM parameters through rescaled triangles



(small but non squashed)
B_D-meson triangle (bd)

$$\frac{V_{ud}V^*_{ub}}{V_{cd}V^*_{cb}} + \frac{V_{td}V^*_{tb}}{V_{cd}V^*_{cb}} + 1 = 0$$

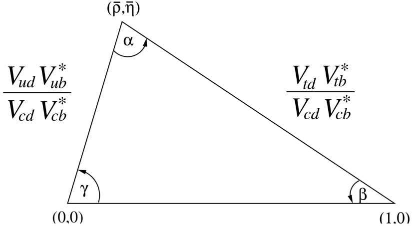


(large but squashed)
D-meson triangle (cu)

$$\frac{V_{ud}V^*_{cd}}{V_{us}V^*_{cs}} + \frac{V_{ub}V^*_{cb}}{V_{us}V^*_{cs}} + 1 = 0$$

"The" unitarity triangle

In practice, rescaled B_d unitarity triangle often used as representation



- good representation of CP-violation (small but non-squashed)
- CKM matrix elements involved in interpretation of B decays
- apex yields two of the four Wolfenstein parameters

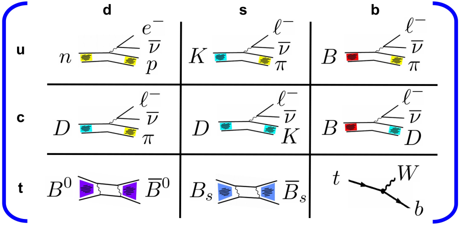
$$\lambda^2 = \frac{|V_{us}|^2}{|V_{ud}|^2 + |V_{us}|^2}, \quad A^2\lambda^4 = \frac{|V_{cb}|^2}{|V_{ud}|^2 + |V_{us}|^2}, \quad \bar\rho + i\bar\eta = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$$

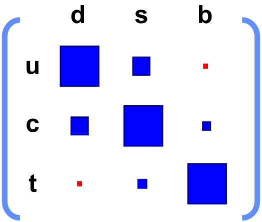
defined in a convention-independent manner

A handle on the CKM matrix

Measurements in terms of hadrons, not of quarks !

V =

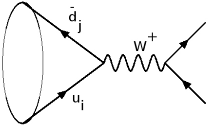


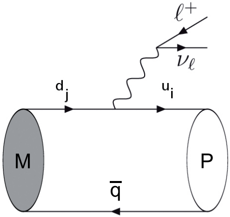


- $d \to u$: Nuclear physics (superallowed β decays)
- $s \to u$: Kaon physics (KLOE, KTeV, NA62)
- $c \to d, s$: Charm physics (CLEO-c, Babar, Belle, BESIII)
- $b \to u, c$ and $t \to d, s$: B physics (Babar, Belle, CDF, DØ, LHCb)
- $t \to b$: Top physics (CDF/DØ, ATLAS, CMS)

How to determine the structure of CKM matrix ?

$|V_{ij}|$ from $\Delta F = 1$





- Leptonic, with f_M decay constant

$$B[M \to \ell\nu_\ell]_{\rm SM} = \frac{G_F^2 m_M m_\ell^2}{8\pi}\left(1 - \frac{m_\ell^2}{m_M^2}\right)^2 |V_{q_u q_d}|^2 f_M^2 \tau_M (1 + \delta_{em}^{M\ell 2})$$

- Semileptonic, with 2 form factors f_+ and f_0

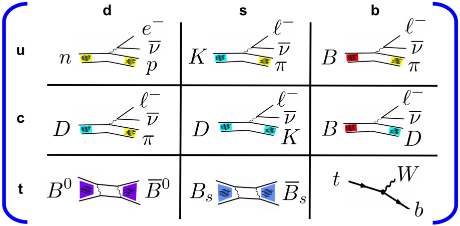
$$\frac{d\Gamma(M \to P\ell\nu)}{dq^2} = \frac{G_F^2 |V_{q_u q_d}|^2}{24\pi^3} \frac{(q^2 - m_\ell^2)^2 \sqrt{E_P^2 - m_P^2}}{q^4 m_H^2}$$
$$\times \left[\left(1 + \frac{m_\ell^2}{2q^2}\right) m_M^2 (E_P^2 - m_P^2) |f_+(q^2)|^2 + \frac{3m_\ell^2}{8q^2}(m_M^2 - m_P^2)^2 |f_0(q^2)|^2\right]$$

- Hadronic quantities, determined from lattice QCD simulations

$$\langle 0|\bar{q}_u \gamma_\mu \gamma_5 q_d|M\rangle \propto f_M \qquad \langle P|\bar{q}_u \gamma_\mu q_d|M\rangle \propto f_+, f_0$$

A few decays of interest

V =

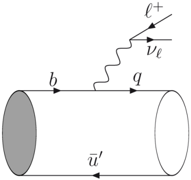


	Leptonic	Semileptonic	Others
$\|V_{ud}\|$	$\pi \to \ell\nu_\ell, \tau \to \pi\nu_\tau$	$\pi^+ \to \pi^0 e^+ \nu_e$	nuclear β decays, n lifetime
$\|V_{us}\|$	$K \to \ell\nu_\ell, \tau \to K\nu_\tau$	$K \to \pi\ell\nu$	inclusive τ decays
$\|V_{cd}\|$	$D^+ \to \ell\nu_\ell$	$D \to \pi\ell\nu_\ell$	μ production by ν beams
$\|V_{cs}\|$	$D_s \to \ell\nu_\ell$	$D \to K\ell\nu_\ell$	$W \to c\bar{s}$
$\|V_{ub}\|$	$B \to \tau\nu$	$B \to \pi\ell\nu_\ell$	$B \to X_u\ell\nu_\ell$ (incl)
$\|V_{cb}\|$	$(B_c \to \tau\nu_\tau)$	$B \to D(^*)\ell\nu$	$B \to X_c\ell\nu_\ell$ (incl)
$\|V_{tb}\|$	–	–	$t \to Wb$

- No direct handle on V_{td}, V_{ts} through tree processes
- Some processes not competitive theo/exp accuracy

arg(V_{ij}) from CP-asymmetries

Take processes conjugate under CP



$$b \to u \quad : \quad A(\bar{B}^0 \to \pi^+ \ell^- \bar{\nu}) \propto V_{ub} \times F_{B\to\pi}$$
$$\bar{b} \to \bar{u} \quad : \quad A(B^0 \to \pi^- \ell^+ \nu) \propto V_{ub}^* \times F_{B\to\pi}$$

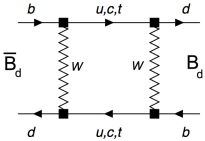
where $F_{B\to\pi}$ form factor encoding hadronisation of quarks into hadrons

General feature : flavour processes with

- weak part : odd under CP (phase from CKM)
- strong part : even under CP (phase from strong interaction)

- $|V_{ij}|$ via CP-conserving quantity $(|A|^2)$
  from rates where hadronic quantities are crucial
- arg V_{ij} via CP-violating quantity $(\mathrm{Re}(A_1 A_2^*), \mathrm{Im}(A_1 A_2^*))$
  from asymmetries where hadronic quantities may cancel out
  $\Longrightarrow$CP-viol. from relative phases between conjugate proc.

CKM elements from $\Delta F = 2$



Loops allow $\Delta F = 2$ FCNC

$\Longrightarrow$ neutral-meson mixing

$$i\frac{d}{dt}\begin{pmatrix} |M(t)\rangle \\ |\bar{M}(t)\rangle \end{pmatrix} = \left(M - \frac{i}{2}\Gamma\right)\begin{pmatrix} |M(t)\rangle \\ |\bar{M}(t)\rangle \end{pmatrix}$$

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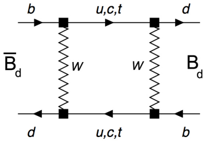

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Diagonalisation: physical $|M_{H,L}\rangle$ of masses $M_{H,L}$, widths $\Gamma_{H,L}$

$$|M_L\rangle = p|M\rangle + q|\bar{M}\rangle, \qquad |M_H\rangle = p|M\rangle - q|\bar{M}\rangle \qquad |p|^2 + |q|^2 = 1$$

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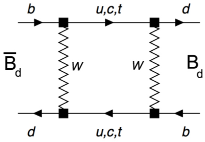

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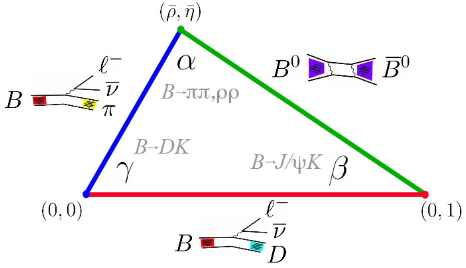
For B_d and B_s dominated by top boxes

$$A_{\Delta B=2} \propto (V_{tb}^* V_{tq})^2 \frac{g^4 m_t^2}{16\pi^2 m_W^4} \langle \bar{B}_q | (\bar{b}_L \gamma_\mu d_L)^2 | B_q \rangle + \ldots$$

- mass difference Δm_q through hadronic contrib $\langle \bar{B}_q | (\bar{b}_L \gamma_\mu d_L)^2 | B_q \rangle$ (bag parameter B_{B_q})
- mixing involve single weak phase: $q/p = \exp[i \arg[(V_{tb}^* V_{tq})^2]$
- similar but more complicated for K (charm and top)

A few modes of interest



Exp. uncertainties		(Controlled) th. uncertainties	
$B \to \pi\pi, \rho\rho$	α	$B(b) \to D(c)\ell\nu$	$\|V_{cb}\|$ vs form factor (OPE)
$B \to DK$	γ	$B(b) \to \pi(u)\ell\nu$	$\|V_{ub}\|$ vs form factor (OPE)
		$M \to \ell\nu(\gamma)$	$\|V_{UD}\|$ vs f_M (decay cst)
$B \to J/\Psi K_s$	β	ϵ_K	$(\bar\rho, \bar\eta)$ vs B_K (bag parameter)
$B_s \to J/\Psi\phi$	β_s	$B_d\bar{B}_d, B_s\bar{B}_s$ mix	$\|V_{tb}V_{tq}\|$ vs $f_B^2 B_B$ (bag param)

- braching ratios of leptonic/semileptonic decays (moduli)
- CP-asymmetries (angles of unitarity triangles(s))
- neutral-meson mixing (product of CKM matrix elements)

Inputs for Summer 18 global fit

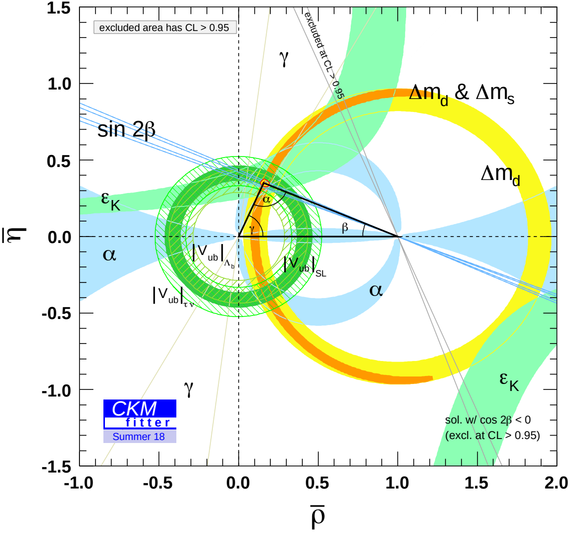
CKMfitter — frequentist ($\simeq \chi^2$ minim.) + Rfit scheme for theory uncert.

data = weak $\otimes$ QCD $\Longrightarrow$ Need for hadronic inputs (mostly lattice)

$\|V_{ud}\|$	superallowed β decays	Towner and Hardy
$\|V_{us}\|$	$K_{\ell 3}$ PDG	$f_+(0) = 0.9661 \pm 0.0014 \pm 0.0022$
	$K \to \ell\nu, \tau \to K\nu_\tau$ PDG	$f_K = 155.6 \pm 0.2 \pm 0.6$ MeV
$\|V_{us}/V_{ud}\|$	$K \to \ell\nu/\pi \to \ell\nu, \tau \to K\nu_\tau/\tau \to \pi\nu_\tau$	$f_K/f_\pi = 1.1959 \pm 0.0007 \pm 0.0029$
ϵ_K	PDG	$\hat{B}_K = 0.7567 \pm 0.0021 \pm 0.0123$
$\|V_{cd}\|$	$D \to \mu\nu$, $D \to \pi\ell\nu$	$f_{D_s}/f_D = 1.175 \pm 0.001 \pm 0.004$, $f_+^{D\to\pi}(0)$
$\|V_{cs}\|$	$D_s \to \mu\nu$, $D_s \to \tau\nu$, $D \to \pi\ell\nu$	$f_{D_s} = 247.8 \pm 0.3 \pm 2.0$ MeV, $f_+^{D\to K}(0)$
$\|V_{ub}\|$	inclusive and exclusive B semileptonic	$\|V_{ub}\| \cdot 10^3 = 3.98 \pm 0.08 \pm 0.22$
$\|V_{cb}\|$	inclusive and exclusive B semileptonic	$\|V_{cb}\| \cdot 10^3 = 41.8 \pm 0.4 \pm 0.6$
$B \to \tau\nu$	$(1.08 \pm 0.21) \cdot 10^{-4}$	$f_{B_s}/f_{B_d} = 1.205 \pm 0.003 \pm 0.006$
		$f_{B_s} = 226.0 \pm 1.3 \pm 2.0$ MeV
$\|V_{ub}/V_{cb}\|$	Λ_b semileptonic decays	integrals of Λ_b form factors
Δm_d	last WA B_d-$\bar{B}_d$ mixing	$B_{B_s}/B_{B_d} = 1.007 \pm 0.013 \pm 0.014$
Δm_s	last WA B_s-$\bar{B}_s$ mixing	$B_{B_s} = 1.327 \pm 0.016 \pm 0.030$
β	last WA $(c\bar{c})$ $K^{(*)}$	no penguin pollution
α	last WA $\pi\pi, \rho\pi, \rho\rho$	isospin
γ	last WA $B \to D^{(*)}K^{(*)}$	GLW/ADS/GGSZ

as well as inputs on $m_t, m_c, \alpha_s(M_Z)$

The current status of CKM



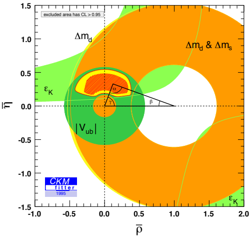
$|V_{ud}|, |V_{us}|, |V_{cb}|, |V_{ub}|_{SL}$

$B \to \tau\nu$

$\Delta m_d, \Delta m_s, \epsilon_K$

$\alpha, \sin 2\beta, \gamma$

$$A = 0.840^{+0.005}_{-0.020}$$
$$\lambda = 0.2247^{+0.0003}_{-0.0001}$$
$$\bar\rho = 0.158^{+0.010}_{-0.007}$$
$$\bar\eta = 0.349^{+0.010}_{-0.007}$$
(68% CL)



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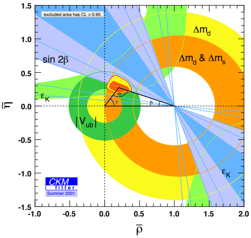

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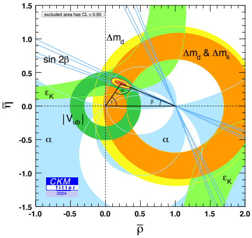

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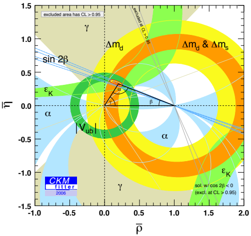
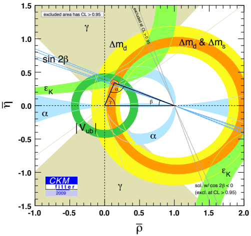
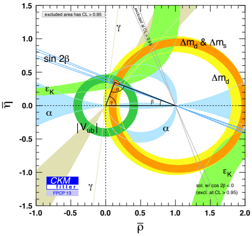


2006



2009



2013

Statistics, or reaching for the optmimum

The name of the game

In these plots, we combine

- many different observables (experimental data)
- which depend on CKM parameters $A, \lambda, \bar{\rho}, \bar{\eta}$
- but also hadronic parameters $f_B, F_{B\to\pi}, B_{B_s} \ldots$

to constrain the value of the CKM parameters

Require a statistical approach

- Bayesian: treat probabilities as (subjective) degree of belief rather than outcome of repeated experiments
- Frequentist: devise methods that will provide values that would be "often" correct if experiments repeated

together with specific treatment of theory uncertainties (hadronic)

A simple case

- Imagine that
  - we measure the observable $X = X_0 \pm \sigma$
  - according to our theory, $X = x(\mu)$ with μ a fundamental parameter
- We want to test a hypothesis $\mathcal{H}_\mu : \mu_t = \mu$

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  $T(X_0;\mu)$ useful to determine if actual data X_0 supports $\mathcal{H}_\mu$
  provided that we know the distribution of $T(X;\mu)$
- p-value defined as $p(X_0;\mu) = \mathcal{P}[T > T(X_0;\mu)]$
  - assuming $\mathcal{H}_\mu$ and repeating the experiment,
    how often would I get T worse than the one observed ?
  - a small p-value indicates that T is rarely larger than $T(X_0;\mu)$
    corresponding to the case where X_0 disfavours $\mathcal{H}_\mu$
  - can be used to build confidence intervals

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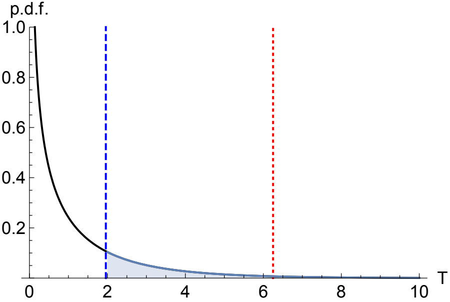

- red: $\mu = 2.5$
- blue: $\mu = -1.4$
- gray: area to be integrated over to get $p(X_0 = 0;\mu = -1.4)$

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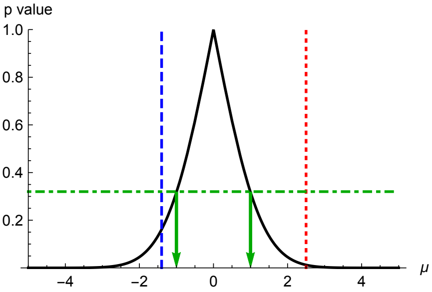
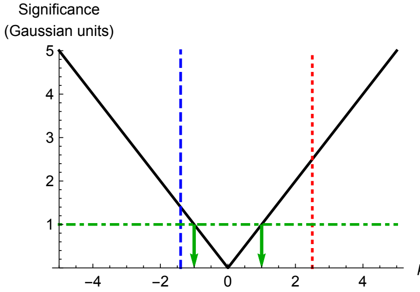
Once p-value is known as a function of μ

- confidence interval at α corresponding to interval with $p = 1-\alpha$





Statistical significance and coverage

- if *p*-value well designed (exact coverage), this random variable has a uniform p.d.f., i.e. for any α, we have $\mathcal{P}[p \leq \alpha | \mathcal{H}_\mu] = \alpha$ $\Longrightarrow$what is needed to defined meaningful confidence intervals !

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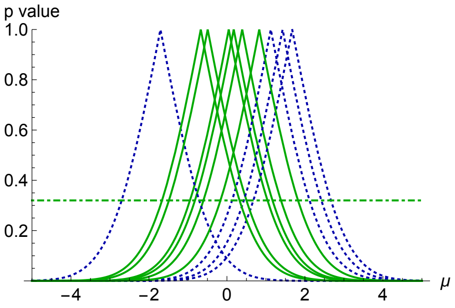

- Assume $\mu_t = 0$ and repeat measuring X with uncertainty $\sigma = 1$
- For each measurement X_0, *p*-value centered around X_0, and each time 68% CI
- If exact coverage, CI contain true value 68% of the time (green curves)

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- Composite hypothesis
  - only some of the theoretical parameters μ fixed explicitly
  - the others, ν, are not determined explicitly [nuisance parameters]
  - by analogy with simple case, Maximal Likelihood Ratio (MLR)

$$T(X;\mu) = -2\log\frac{\max_{\nu'}\mathcal{L}_X(\mu,\nu')}{\max_{\mu',\nu'}\mathcal{L}_X(\mu',\nu')}$$

  - empirically powerful, but no general proof
  - Wilks' theorem: in large-sample limit, under regularity conditions, T distributed as χ^2 with dim given by the number of params tested

Applying Maximal Likelihood Ratio

Test statistic

- one or two parameters of interest, and remaining nuisance params
  for instance $\mu = (\bar{\rho}, \bar{\eta}) \qquad \nu = (A, \lambda, f_B, F^{K\to\pi}, B_{B_s} \ldots)$
- test statistic from the likelihoods

$$T(X;\mu) = -2\log\frac{\max_{\nu'}\mathcal{L}_X(\mu,\nu')}{\max_{\mu',\nu'}\mathcal{L}_X(\mu',\nu')} = \chi^2(\mu) - \min_\mu \chi^2(\mu) = \Delta\chi^2$$

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- $T = \Delta\chi^2$ as χ^2-law with N_{dof} yields p-value as a function of μ to determine confidence intervals/regions on μ
- $\min_\mu \chi^2(\mu) = \chi^2_{\min}$ as indication of overall goodness of fit
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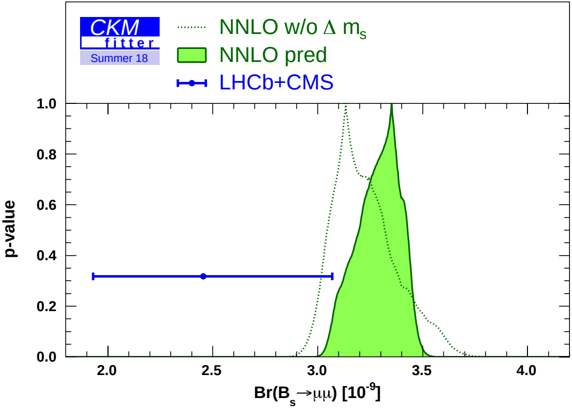
$\equiv$ least squares and confidence intervals from $\Delta\chi^2$ if Gaussian

A typical outcome: $Br(B_s \to \mu\mu)$

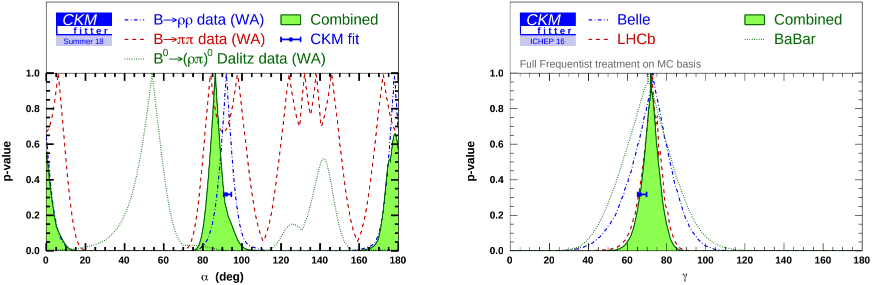


- many different inputs constraining the value of CKM parameters
- out of which a p-value curve can be shown for $Br(B_s \to \mu\mu)$
- best-fit point for $p = 1$, 68% CI at $p = 0.32$, 95% CI at $p = 0.05$
- comparison with experimental value (in blue)

Special cases

Angles deserving a special statistical treatment due to their extraction

- α: discrete ambiguities at the level of the measurement
- γ: bias depending on the size of hadronic contributions, altering the coverage and requiring specific determination of *p*-values



Not Gaussian, described through a Look-Up Table (LUT) file

General expressions for special cases

If Wilks' theorem does not apply, no simple analytic expressions

- p.d.f. for measurement of obs X under hypothesis $\mathcal{H}_\mu$
  $g(X;\mu) = \mathcal{L}_X(\mu)$ defining the likelihood
- test statistic in terms of likelihoods

$$T(X;\mu) = -2\log \frac{\max_{\nu'} \mathcal{L}_X(\mu,\nu')}{\max_{\mu',\nu'} \mathcal{L}_X(\mu',\nu')}$$

- p.d.f. for test statisttic

$$h(T|\mathcal{H}_\mu) = \int dX\, \delta\left[T - T(X;\mu)\right] g(X;\mu)$$

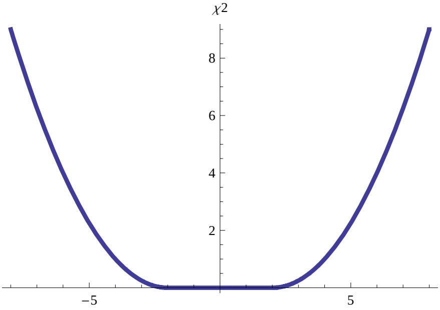
- p-value for μ if X_0 is measured, and corresponding CI

$$1 - p(X_0;\mu) = \int_0^{T(X_0;\mu)} dT\, h(T|\mathcal{H}_\mu) = \mathcal{P}[T < T(X_0;\mu)]$$

p-value can thus be computed numerically (Toy Monte Carlo), but only used if away from asymptotic limit (no Wilks' theorem)

Theoretical uncertainties

- Observable = CKM $\otimes$ hadronic
- hadronic input often from lattice QCD simulations: $X = X_0 \pm \sigma \pm \Delta$
  - σ statistical, scales with size of sampling, Gaussian model
  - Δ theoretical, dominant for lattice, modelling with no consensus

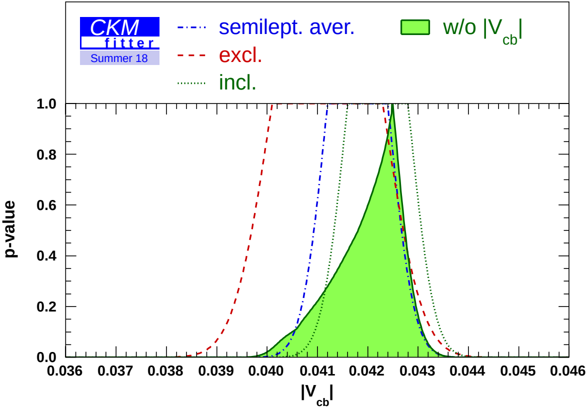


- CKMfitter: Rfit approach
  - modify likelihood $\mathcal{L} = \exp(-\chi^2/2)$
  - χ^2 with flat bottom (theo/syst) and parabolic walls (stat)
  - all values within range of syst treated on same footing
  - averaging procedure designed consistently

- Other approaches: Gaussian (combined in quadrature with statistics), adaptive…

[Charles et al.]

Another typical outcome: $|V_{cb}|$



- Inclusive ($B \to X_c \ell\nu$) and exclusive ($B \to D(^*)\ell\nu$) determinations with significant theoretical uncertainties (flat top of p-values)
- Average designed to take into account Rfit for theo uncertainties
- Global fit prediction (without $|V_{cb}|$ input) smooth

Take-home message

$p = (A, \lambda, \bar{\rho}, \bar{\eta} \ldots) = (q, r)$

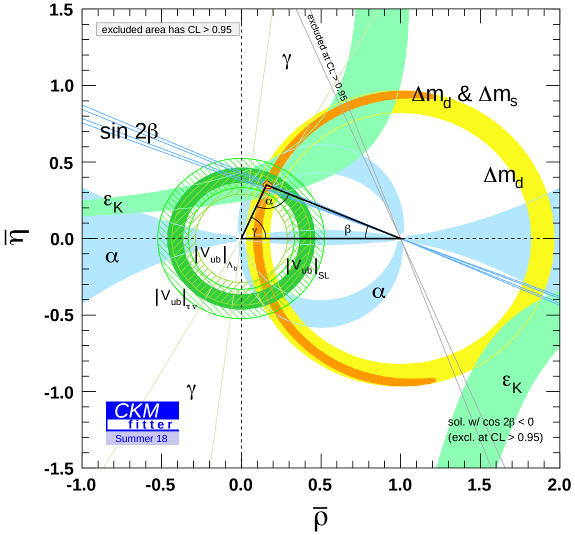
- q parameters of interest (CKM), r nuisance parameters (hadronic)
- $\mathcal{O}_{\rm meas} \pm \sigma_{\mathcal{O}}$ experimental values of observables
- $\mathcal{O}_{\rm th}(p)$ theoretical description in a given model

$$\mathcal{L}(p) = \prod_{\mathcal{O}} \mathcal{L}_{\mathcal{O}}(p) \qquad T(p) = -2 \ln \mathcal{L}(p) = \sum_{\mathcal{O}} \left(\frac{\mathcal{O}_{\rm th}(p) - \mathcal{O}_{\rm meas}}{\sigma_{\mathcal{O}}} \right)^2$$

$$\chi^2(q) = \min_r T(q, r)$$

- Central value: estimator $\hat{q}$ max likelihood $\quad \chi^2(\hat{q}) = \min_q \chi^2(q)$
- Range: confidence level (p-value) for q_0 computed from $\Delta\chi^2(q_0) = \chi^2(q_0) - \min_q \chi^2(q)$, assuming χ^2 law with $N = dim(q)$
- Specific (Rfit) treatment of theoretical uncertainties modifying $\mathcal{L}$, and impacting the procedure to average measurements

The current status of CKM



$|V_{ud}|, |V_{us}|, |V_{cb}|, |V_{ub}|_{SL}$

$B \to \tau\nu$

$\Delta m_d, \Delta m_s, \epsilon_K$

$\alpha, \sin 2\beta, \gamma$

$$A = 0.840^{+0.005}_{-0.020}$$
$$\lambda = 0.2247^{+0.0003}_{-0.0001}$$
$$\bar\rho = 0.158^{+0.010}_{-0.007}$$
$$\bar\eta = 0.349^{+0.010}_{-0.007}$$
(68% CL)

Any questions ?