A lattice QCD calculation of the hadronic light-by-light contribution to the magnetic moment of the muon

Harvey Meyer
J. Gutenberg University Mainz

Muon g-2 Theory Initiative workshop in memoriam Simon Eidelman,
2 July 2021 (virtual format)

Cluster of Excellence

THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

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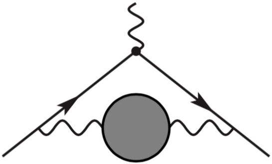
First idea to compute a_μ^{HLbL} in lattice QCD:
Hayakawa, Blum, Izubuchi, Yamada [hep-lat/0509016]

Contributions to formalism for the Mainz approach (since 2014):
N. Asmussen, E.-H. Chao, A. Gérardin, J. Green, J. Hudspith, HM, A. Nyffeler

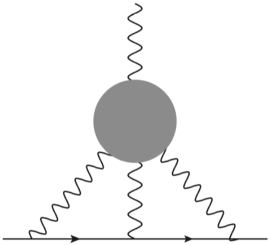
This talk is mostly based on

1. Chao, Gérardin, Green, Hudspith, HM, Eur.Phys.J.C 80 (2020) 9, 869 [2006.16224]
2. Chao, Hudspith, Gérardin, Green, HM, Ottnad, [2104.02632].

Source of dominant uncertainties in SM prediction for $(g-2)_\mu$



Hadronic vacuum polarisation



Hadronic light-by-light scattering

HVP: $O(\alpha^2)$, about $7000 \cdot 10^{-11}$

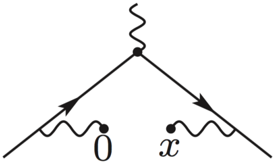
$\Rightarrow$ target accuracy: $\lesssim 0.5\%$

HLbL: $O(\alpha^3)$, about $100 \cdot 10^{-11}$

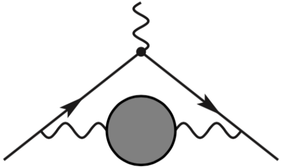
$\Rightarrow$ target accuracy: $\lesssim 15\%$.

Recall: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (251 \pm 59) \cdot 10^{-11}$.

Analogy: hadronic vacuum polarization in x-space [HM 1706.01139]



$\Rightarrow$



QED kernel $H_{\mu\nu}(x)$ $\qquad$ $a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x\; H_{\mu\nu}(x)\,\Big\langle j_\mu(x) j_\nu(0)\Big\rangle_{\rm QCD},$$

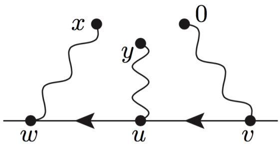
$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

Weight functions $\mathcal{H}_i$ known in terms of Meijer's functions.

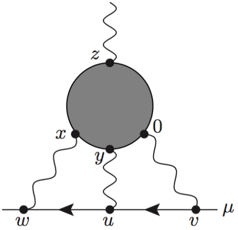
Due to $\partial_\mu j_\mu = 0$, freedom to add to $H_{\mu\nu}$ terms like $\partial_\mu\partial_\nu f(|x|)$ or $\partial_\mu(x_\nu f(|x|)$ for f's that do not generate boundary terms upon partial integration [Cè et al, 1811.08669].

Coordinate-space approach to a_μ^{HLbL}, Mainz version

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



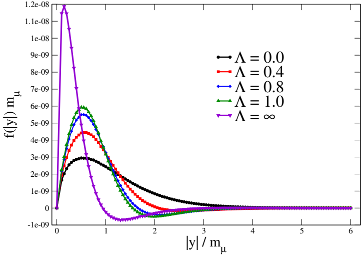
$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \left[\int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{=\mathrm{QCD\ blob}} \right].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \, z_\rho \left\langle j_\mu(x) \, j_\nu(y) \, j_\sigma(z) \, j_\lambda(0) \right\rangle.$$

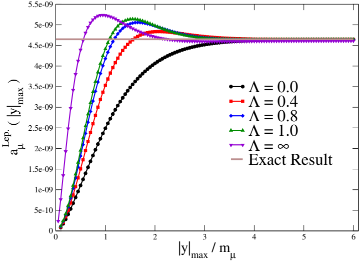
- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects & only a 1d integral to sample the integrand in $|y|$.

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384, 1609.08454]

Tests of the framework and adjustments to the kernel



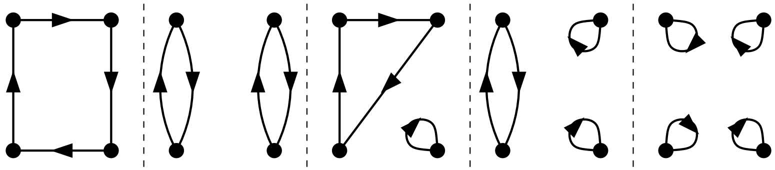
Integrands (Lepton loop, method 2)



Correspondings integrals

- The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six 'weight' functions of the variables $(x^2, x\cdot y, y^2)$.
-
$$\bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \partial^{(x)}_\mu(x_\alpha e^{-\Lambda m_\mu^2 x^2/2})\bar{\mathcal{L}}_{[\rho,\sigma];\alpha\nu\lambda}(0,y) - \partial^{(y)}_\nu(y_\alpha e^{-\Lambda m_\mu^2 y^2/2})\bar{\mathcal{L}}_{[\rho,\sigma];\mu\alpha\lambda}(x,0),$$
- Using this kernel, we have reproduced (at the 1% level) known results for a range of masses for:
  1. the lepton loop (spinor QED, shown in the two plots);
  2. the charged pion loop (scalar QED);
  3. the π^0 exchange with a VMD-parametrized transition form factor.

Wick-contraction topologies in HLbL amplitude $\langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle$



First two classes of diagrams turn out to be dominant, with a cancellation between them.

Example: $\Pi = \langle (j_u - j_d)(j_u - j_d)(j_u - j_d)(j_u - j_d)\rangle$ does not contain the π^0 pole (π^0 only couples to one isovector, one isoscalar current).

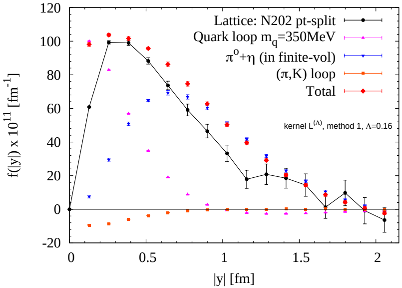
Write out the Wick contractions: $\Pi = 2 \cdot \Pi^{(4)} + 4 \cdot \Pi^{(2,2)}$

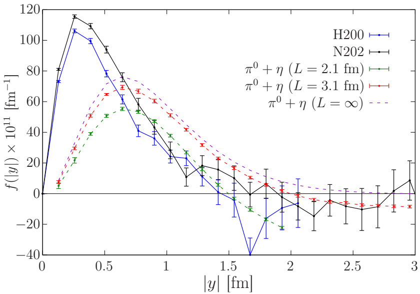
In kinematic regime where π^0 dominates: $|\Pi| \ll \Pi^{(4)} \Rightarrow \Pi^{(2,2)} \approx -\frac{1}{2}\Pi^{(4)}$.

Including charge factors: $\Big[(Q_u^2 + Q_d^2)^2 \Pi^{(2,2)}\Big] = -\frac{25}{34}\Big[(Q_u^4 + Q_d^4)\Pi^{(4)}\Big]$.

Large-N_c argument by J. Bijnens, 1608.01454; see also 1712.00421.

Quark-connected integrand at $m_\pi = m_K \simeq 415\,\text{MeV}$



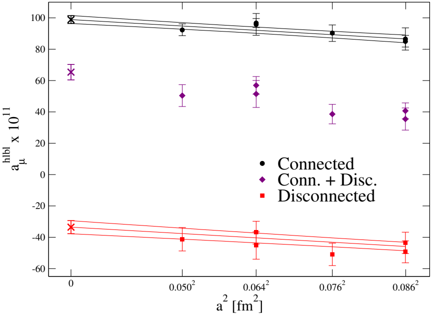


- Partial success in understanding the integrand in terms of familiar hadronic contributions.
- Reasonable understanding of magnitude of finite-size effects. ($L_{\rm H200} = 2.1\,\text{fm}$, $L_{\rm N202} = 3.1\,\text{fm}$)

2006.16224 Chao et al. (EPJC)

$a_\mu^{\rm HLbL}$ at $m_\pi = m_K \simeq 415$ MeV

[Chao, Gérardin, Green, Hudspith, HM 2006.16224 (EPJC)]



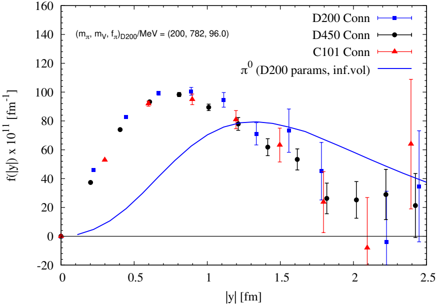
$$a_\mu^{\rm hlbl,SU(3)_f} = (65.4 \pm 4.9 \pm 6.6) \times 10^{-11}.$$

Guesstimating the result at physical quark masses: correct for π^0 exchange

$$a_\mu^{\rm hlbl,SU(3)_f} - a_\mu^{{\rm hlbl},\pi^0,{\rm SU(3)_f}} + a_\mu^{{\rm hlbl},\pi^0,{\rm phys}} = (104.1 \pm 9.1_{\rm stat}) \times 10^{-11}.$$

Estimate based on lattice QCD calculation of $\pi^0 \to \gamma^*\gamma^*$ transition form factor [Gérardin, HM, Nyffeler 1903.09471 (PRD)].

Integrand of connected contribution at $m_\pi \approx 200$ MeV

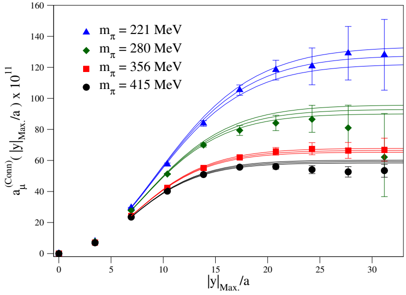


- using four local vector currents
- based on 'Method 2'.

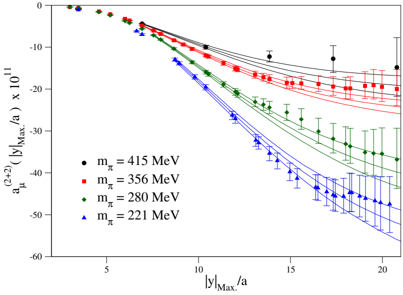
En-Hung Chao, Renwick Hudspith, Antoine Gérardin, Jeremy Green, HM, Konstantin Ottnad
2104.02632

Truncated integral for a_μ^{HLbL}

Connected



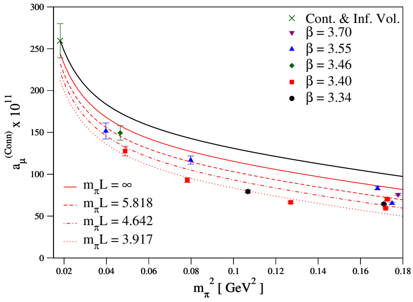
(2+2) Disconnected



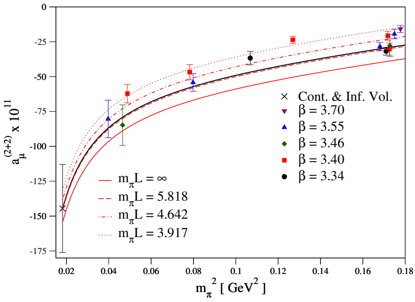
- Extend reach of the signal by two-param. fit $f(y) = A|y|^3 \exp(-M|y|)$;
- provides an excellent description of the π^0 exchange contribution in infinite volume.
- We see a clear increase of the magnitude of both connected and disconnected contributions.

Chiral, continuum, volume extrapolation

Connected contribution

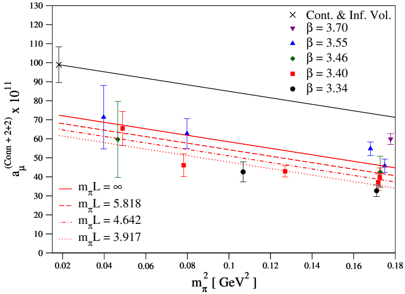


disconnected contribution

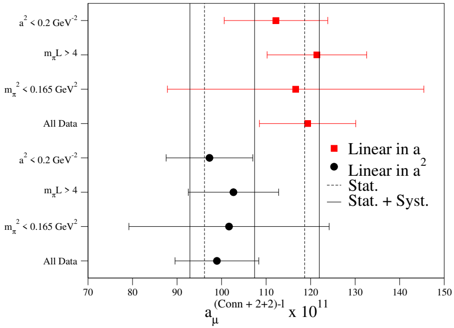


Total light-quark contribution:

- vol. dependence: $\propto \exp(-m_\pi L/2)$
- pion-mass dependence fairly mild (!)



Extrapolating the sum of conn. & disconn.



$$\text{Ansatz}: \quad Ae^{-m_\pi L/2} + Ba^2 + D + Em_\pi^2$$

- results very stable with respects to cuts in a, m_π or $m_\pi L$.
- largest systematic comes from choice of continuum limit ansatz.
- final result: central value from fitting these results with a constant; systematic error set to $\sqrt{(1/N)\sum_{i=1}^{N}(y_i - \bar{y})^2}$ as a measure of the spread of the results.

Overview table

Contribution	Value$\times 10^{11}$
Light-quark fully-connected and $(2+2)$	107.4(11.3)(9.2)
Strange-quark fully-connected and $(2+2)$	$-0.6(2.0)$
$(3+1)$	0.0(0.6)
$(2+1+1)$	0.0(0.3)
$(1+1+1+1)$	0.0(0.1)
Total	106.8(14.7)

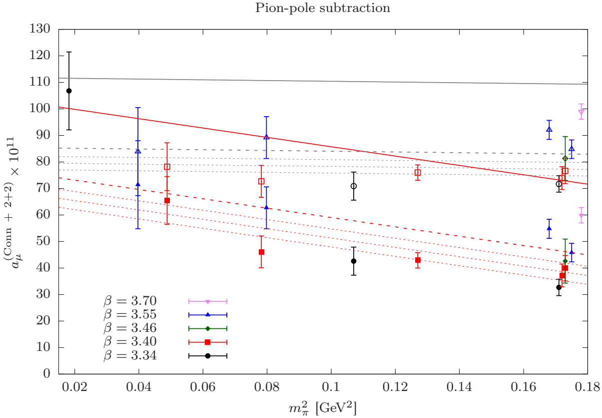
- error dominated by the statistical error and the continuum limit.
- all subleading contributions have been tightly constrained and shown to be negligible; quark loops with a single current insertion generated by K. Ottnad within G. von Hippel's DFG project HI 2048/1-2.

[Chao et al, 2104.02632]

New: subtract out π^0 exchange prior to chiral extrapolation

Attempt at reducing the m_π-dependence:

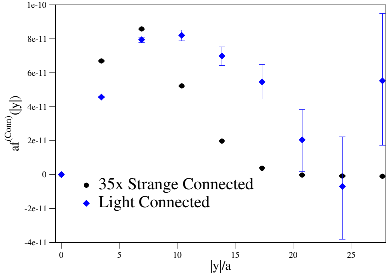
$$a_\mu^{\rm HLbL}(m_\pi) - a_\mu^{{\rm HLbL},\pi^0}(m_\pi) + a_\mu^{{\rm HLbL},\pi^0}(m_\pi^{\rm phys})$$

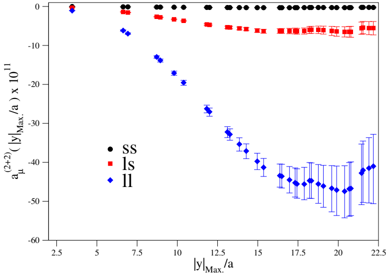


- Used $a_\mu^{{\rm HLbL},\pi^0}(m_\pi^{\rm phys}) = 59.7 \pm 3.6$ from [1903.09471].
- Result of extrapolating linearly in m_π^2 and in a^2: $a_\mu^{\rm HLbL} = 111.8 \pm 11_{\rm stat}$
- to be compared with $106.8 \pm 14.7_{\rm tot}$ (black data pt on figure) [2104.02632].

Strange contribution

Ensemble C101 ($48^3 \times 96$, $a = 0.086$ fm, $m_\pi = 220$ MeV)



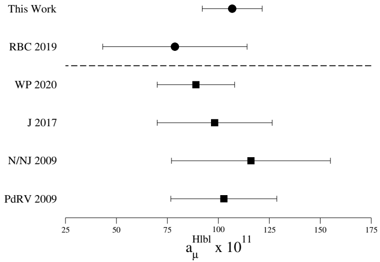


(2,2) disconnected contributions.

NB. Strange integrand has a factor 17 suppression due to charge factor.

The (2,2) sl contribution is practically the correlator $\langle (j_u - j_d)(j_u - j_d) j_s j_s \rangle$.
Challenge: who can predict this correlator?

Conclusion on $a_\mu^{\rm HLbL}$

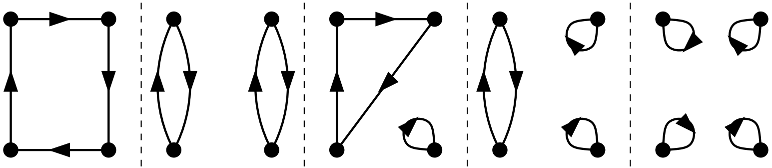


[Fig. from 2104.02632]

- Results from the Bern dispersive framework and from two independent lattice QCD calculations are in good agreement and have comparable uncertainties.
- It is now practically excluded that $a_\mu^{\rm HLbL}$ can by itself explain the tension between the SM prediction and the experimental value of a_μ.

Backup Slides

Wick-contraction topologies in HLbL amplitude $\langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle$

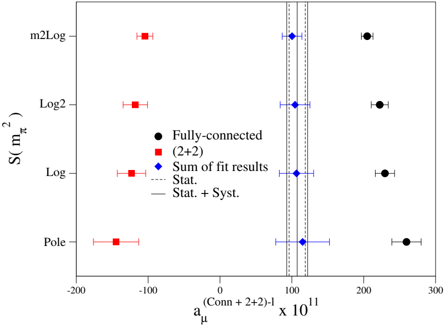


First two classes of diagrams thought to be dominant, with a cancellation between them:

	Weight factor of:	fully connected	(2,2) topology
SU(2)$_\mathrm{f}$:	isovector-meson exchange	$34/9 \approx 3.78$	$-25/9 \approx -2.78$
$m_s = \infty$	isoscalar-meson exchange	0	1
	$\pi^\pm$ loop (-28/81 $\in$ (3,1) topol.)	34/81	75/81
SU(3)$_\mathrm{f}$:	octet-meson exchange	3	-2
$m_s = m_{ud}$	singlet-meson exchange	0	1

Large-N_c argument by J. Bijnens, 1608.01454; see also 1712.00421; Fig. by J. Green.

Separate extrapolation of conn. & disconn.



$$\text{Ansatz}: \quad Ae^{-m_\pi L/2} + Ba^2 + CS(m_\pi^2) + D + Em_\pi^2$$

- chirally singular behaviour cancels in sum of connected and disconnected.

Direct lattice calculation of HLbL in $(g-2)_\mu$

At first, this was thought of as a QED+QCD calculation
[pioneered in Hayakawa et al., hep-lat/0509016].

Today's viewpoint: the calculation is considered a QCD four-point Green's function, to be integrated over with a weighting kernel which contains all the QED parts.

RBC-UKQCD: calculation of a_μ^{HLbL} using coordinate-space method in muon rest-frame; photon+muon propagators:

- either on the $L \times L \times L$ torus (QED$_L$) (1510.07100–present)
- or in infinite volume (QED$_\infty$) (1705.01067–present).

Mainz:

- manifestly covariant QED$_\infty$ coordinate-space approach, averaging over muon momentum using the Gegenbauer polynomial technique (1510.08384–present).

Wisdom gained from model calculations Prades, de Rafael, Vainshtein 0901.0306

- heavy (charm) quark loop makes a small contribution

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \mathcal{Q}_c^4 c_4 \frac{m_\mu^2}{m_c^2} + \dots , \qquad c_4 \approx 0.62.$$

- Light-quarks: (A) charged pion loop is negative, proportional to m_π^{-2}:

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, c_2 \frac{m_\mu^2}{m_\pi^2} + \dots , \qquad c_2 \approx -0.065.$$

  (B) The neutral-pion exchange is positive, $\log^2(m_\pi^{-1})$ divergent:
  Knecht, Nyffeler, Perrottet, de Rafael PRL88 (2002) 071802

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \frac{m_\mu^2}{48\pi^2 (F_\pi^2/N_c)} \Big[\log^2 \frac{m_\rho}{m_\pi} + {\rm O}\Big(\log \frac{m_\rho}{m_\pi} \Big) + {\rm O}(1) \Big].$$

- For real-world quark masses: using form factors for the mesons is essential, and resonances up to 1.5 GeV can still be relevant ⇒ medium-energy QCD.

Method based on QED$_L$ pursued by RBC/UKQCD collaboration

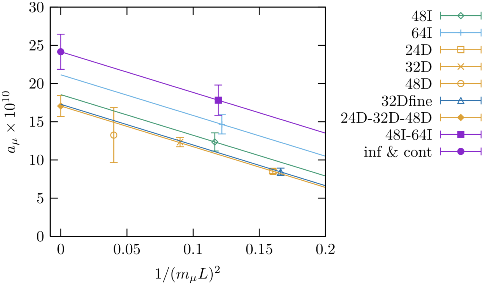
- Photon and muon propagators computed with lattice action.
- Photon $\boldsymbol{q} = 0$ spatial zero-mode removed 'by hand'.
- magnetic moment computed with formula of the type $\boldsymbol{\mu} = \frac{1}{2}\int d^3r\, \boldsymbol{r} \times \boldsymbol{j}$.
- Use domain-wall fermions practically at physical quark masses.
- Gluons: use gauge ensembles with two different actions
- Largest spatial lattice used: 64^3
- Extrapolation of the type

$$a_\mu^{\rm HLbL}(L,a) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3}\Big)\Big(1 - c_1(m_\mu a)^2 + c_2(m_\mu a)^4\Big).$$

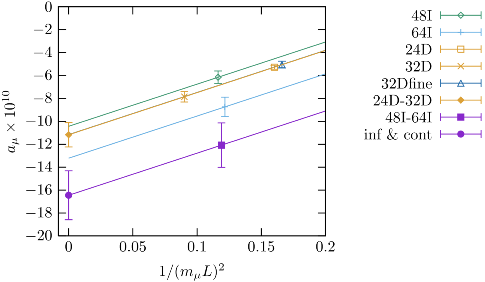
[Blum et al. 1911.08123 (PRL)]

RBC/UKQCD (QED$_L$): final extrapolation [Blum et al. 1911.08123 (PRL)]

Connected $\longrightarrow$



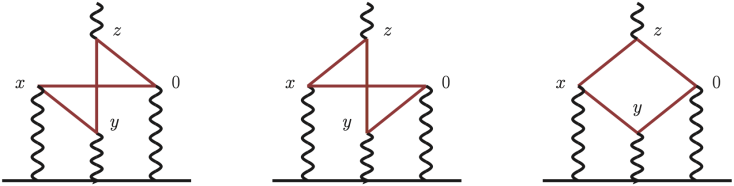
Disconnected $\longrightarrow$



$$\text{Total}: \quad a_\mu^{\text{HLbL}} = (78.7 \pm (30.6)_{\text{stat}} \pm (17.7)_{\text{syst}}) \cdot 10^{-11}.$$

Wick contractions of the connected contribution

Method 1:



$\widetilde{\Pi}^{(1)}_{\mu\nu\sigma\lambda}(x,y,z)$

Method 2:

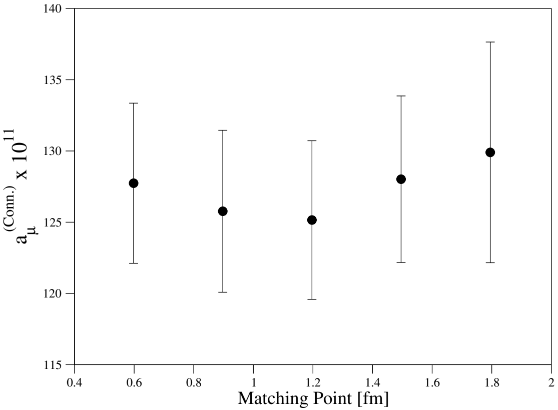
$$a_\mu^{\rm conn} = -\frac{18}{81} Z_{\rm V}^4 \frac{m_\mu e^6}{3} 2\pi^2 \int d|y||y|^3 \int d^4x$$

$$\Bigg((\bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) + \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\nu\mu\lambda}(y,x) - \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y)) \int d^4z \; z_\rho \widetilde{\Pi}^{(1)}_{\mu\nu\sigma\lambda}(x,y,z)$$

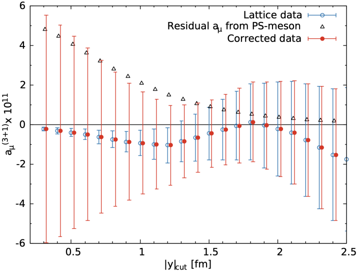
$$+ \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y) x_\rho \int d^4z \; \widetilde{\Pi}^{(1)}_{\mu\nu\sigma\lambda}(x,y,z) \Bigg).$$

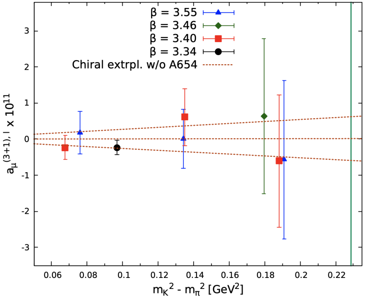
Dependence of connected a_μ^{HLbL} on starting point of using fit

Ensemble C101 ($48^3 \times 96$, $a = 0.086$ fm, $m_\pi = 220$ MeV)



The contribution of the (3+1) topology

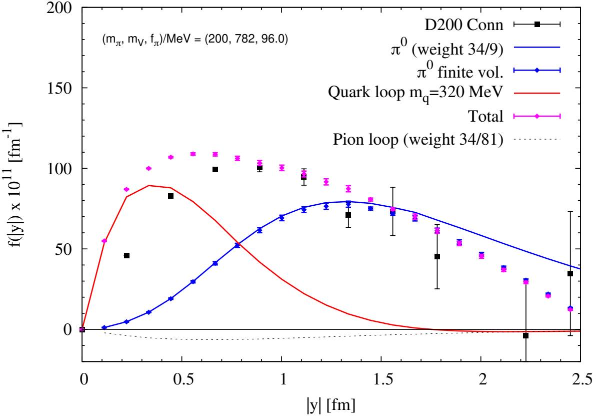




Final result: $a_\mu^{\rm hlbl,3+1} = (0.0 \pm 0.6) \times 10^{-11}$.

Connected integrand for $a_\mu^{\rm HLbL} = \int_0^\infty d|y|\, f(|y|)$ vs. hadronic models

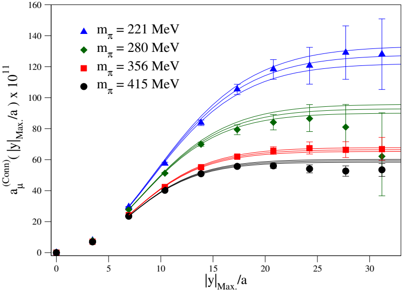
$m_\pi = 200$ MeV, $m_K = 480$ MeV: $(64^3 \times 128$ lattice, $a = 0.064\,$fm)



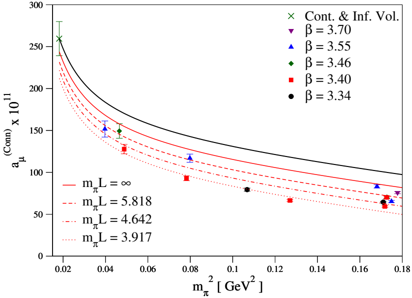
En-Hung Chao, Renwick Hudspith, Antoine Gérardin, Jeremy Green, HM, Konstantin Ottnad
2104.02632

The connected contribution

Cumulated $a_\mu^{\mathrm{HLbL}} = \int_0^{|y|_{\max}} d|y|\, f(|y|)$

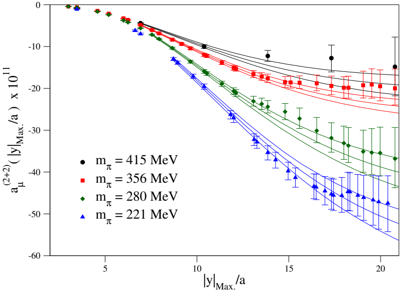


Chiral, continuum, vol. extrapolation

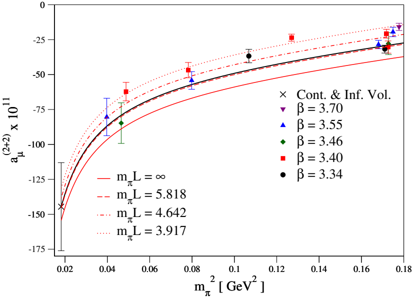


The disconnected contribution

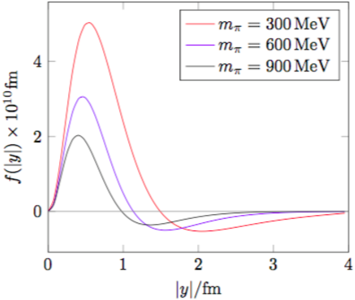
Cumulated $a_\mu^{\mathrm{HLbL}} = \int_0^{|y|_{\max}} d|y|\, f(|y|)$



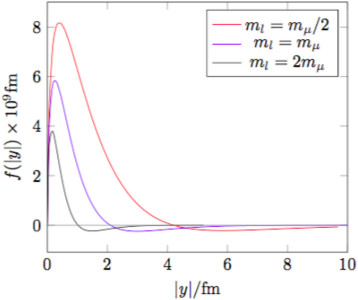
Chiral, continuum, vol. extrapolation



Continuum tests: contribution of the π^0 and lepton loop to a_μ^{HLbL}



Integrand of the pion-pole contribution with VMD transition form factor.

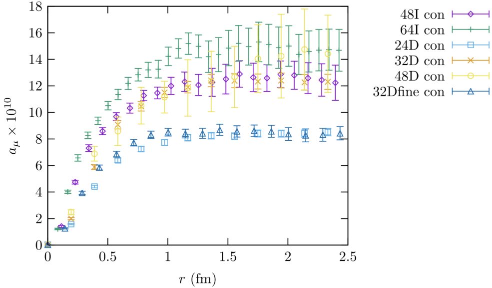


Integrand of the lepton-loop contribution.

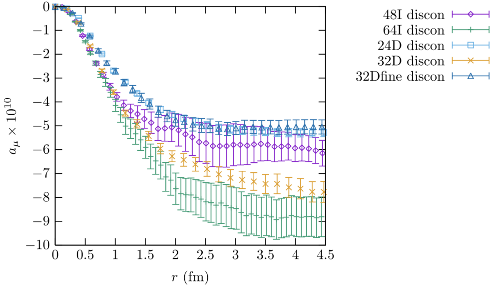
- Even more freedom in choosing best lattice implementation than in HVP.
- The form of the $|y|$-integrand depends on the precise QED kernel used: can perform subtractions (Blum et al. 1705.01067; $\mathcal{L} \to \mathcal{L}^{(2)}$), impose Bose symmetries on $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ or add a longitudinal piece $\partial_\mu^{(x)} f_{\rho;\nu\lambda\sigma}(x,y)$.

RBC/UKQCD (QED_L): cumulative contributions to a_μ^{HLbL}

Connected $\longrightarrow$



Disconnected $\longrightarrow$



[Blum et al. 1911.08123 (PRL)]

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e.g.

$$\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \int_u G_{m_\gamma}(u-y)\Big\langle J(\hat\epsilon,u)J(\hat\epsilon,x-u)\Big\rangle_{\hat\epsilon}, \quad J(\hat\epsilon,y) \equiv \int_u G_0(y-u)\, e^{m\hat\epsilon\cdot u} G_m(u)\\
V_\delta(x,y) &= x_\delta \bar{\mathfrak{g}}^{(1)}(|x|,\hat x\cdot\hat y,|y|) + y_\delta \bar{\mathfrak{g}}^{(2)}(|x|,\hat x\cdot\hat y,|y|),\\
T_{\alpha\beta}(x,y) &= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.