Lattice calculation of the hadronic light-by-light contribution to the muon $g-2$ by the RBC-UKQCD collaborations

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KEK IPNS, High energy physics laboratory in Nagoya University

Outline

1. QED_L: RBC-UKQCD 2020
2. QED_∞: working in progress

HLbL QED_L: RBC-UKQCD 2020

PHYSICAL REVIEW LETTERS **124,** 132002 (2020)

Editors' Suggestion   Featured in Physics

Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

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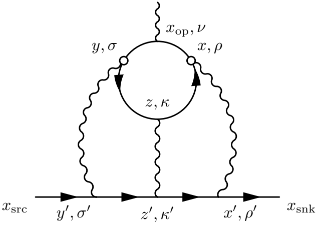
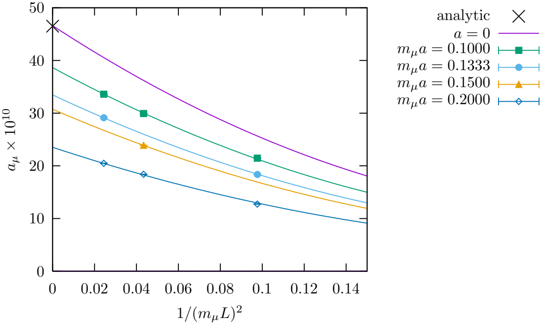
- First lattice result for the hadronic light-by-light scattering contribution to the muon $g - 2$ with all errors systematically controlled.
- Lattice calculation directly at the physical pion mass and no Chiral extrapolation is needed.
- $\mathcal{O}(1/L^2, a^2)$ extrapolation performed to obtain the infinite volume and continuum limit.

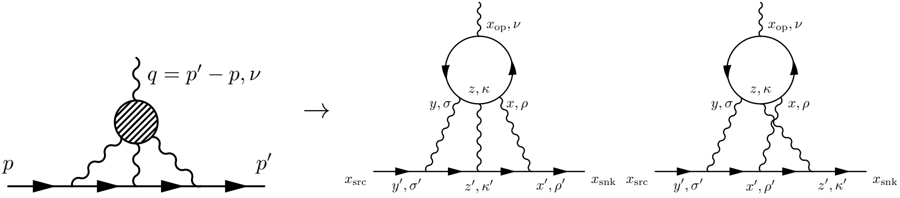
- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.





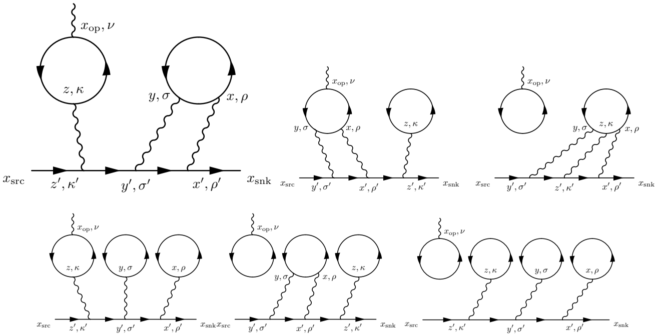
$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2 a^2 + c_2' a^4) \;\rightarrow\; F_2 = 46.6(2)\times 10^{-10} \quad (19)$$

- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

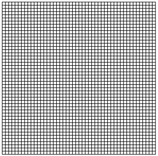


- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional four different permutations of photons not shown.
- The photons can be connected to different quark loops. These are referred to as the disconnected diagrams. They will be discussed later.
- First results are obtained by T. Blum et al. 2015 (PRL 114, 012001).

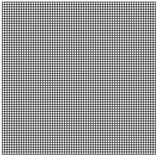
- One diagram (the biggest diagram below) do not vanish even in the SU(3) limit.
- We extend the method and computed this leading disconnected diagram as well.



- Permutations of the three internal photons are not shown.
- **Gluons exchange between and within the quark loops are not drawn.**
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

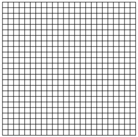


48I: $48^3 \times 96$, $5.5\,\mathrm{fm}$ box

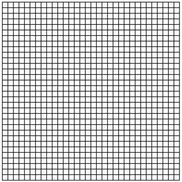


64I: $64^3 \times 128$, $5.5\,\mathrm{fm}$ box

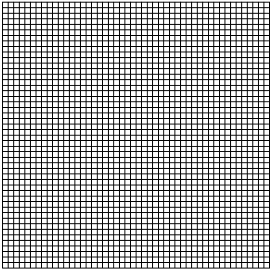
Phys. Rev. D 93, 074505 (2016)



24D: $24^3 \times 64$, $4.8\,\mathrm{fm}$ box



32D: $32^3 \times 64$, $6.4\,\mathrm{fm}$ box

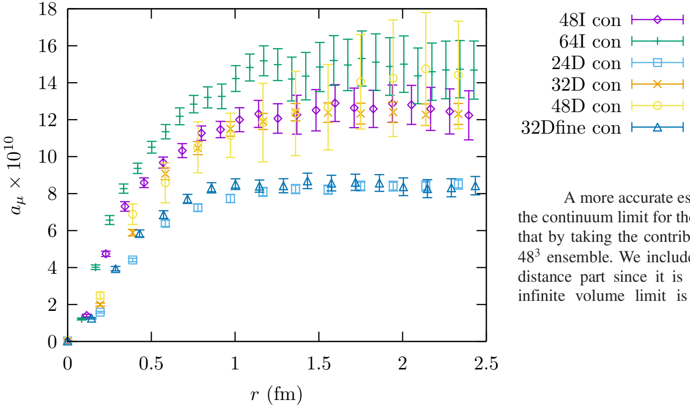


48D: $48^3 \times 64$, $9.6\,\mathrm{fm}$ box

32Dfine: $32^3 \times 64$, $4.8\,\mathrm{fm}$ box

T. Blum et al 2020. (PRL 124, 13, 132002)

$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\rm op})u_s(\vec{0})$$

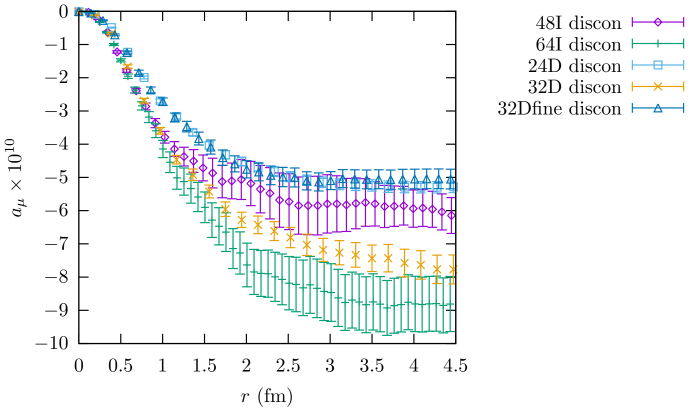


A more accurate estimate can be obtained by taking the continuum limit for the sum up to $r = 1$ fm, and above that by taking the contribution from the relatively precise 48^3 ensemble. We include a systematic error on this long distance part since it is not extrapolated to $a = 0$. The infinite volume limit is taken as before.

Partial sum is plotted above. Full sum is the right most data point.

T. Blum et al 2020. (PRL 124, 13, 132002)

$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\text{op}}}\frac{1}{2}(\vec{x}_{\text{op}}-\vec{x}_{\text{ref}})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\text{op}})u_s(\vec{0})$$

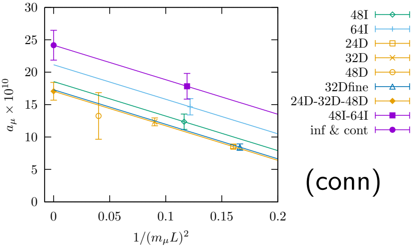


Partial sum is plotted above. Full sum is the right most data point.

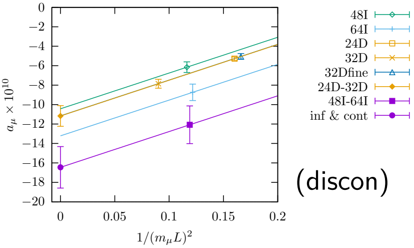
T. Blum et al 2020. (PRL 124, 13, 132002)

HLbL QED_L: Extrapolation

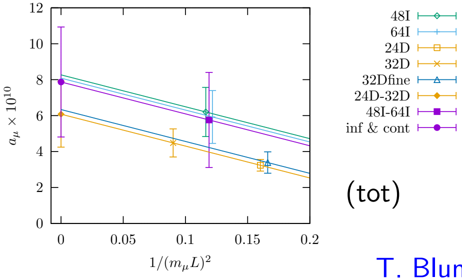
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\right)$$



(conn)



(discon)



(tot)

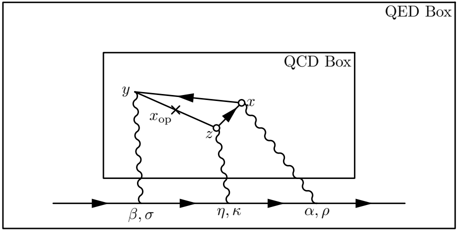
T. Blum et al 2020. (PRL 124, 13, 132002)

	con	discon	tot
a_μ	24.16(2.30)	-16.45(2.13)	7.87(3.06)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.88(0.31)	0.71(0.28)	0.95(0.92)
sys $\mathcal{O}(a^2 \log(a^2))$	0.23(0.08)	0.25(0.09)	0.02(0.11)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.49(1.37)	1.08(1.57)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.11(1.32)	3.99(1.29)	1.77(1.13)

- Same method is used for esimating the systematic error of individual and total contribution.
- Systematic error has some cancellation between the connected and disconnected diagrams.

T. Blum et al 2020. (PRL 124, 13, 132002)

- $a_\mu = 7.87(3.06)_{\text{stat}}(1.77)_{\text{sys}} \times 10^{-10}$.
  T. Blum et al 2020. (PRL 124, 13, 132002)
- Consistent with more recent Mainz group result:
  $a_\mu = 10.68(1.47) \times 10^{-10}$ E. H. Chao et al 2021. (arXiv:2104.02632)
- Consistent with the analytical approach:
  $9.2(1.9) \times 10^{-10}$ (White paper 2020).
- Working on the infinite volume QED approach pioneered by the Mainz group.

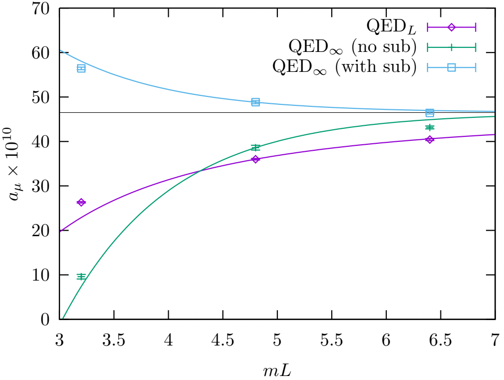


$$
\begin{aligned}
i^3 \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) &= \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \mathfrak{G}_{\kappa,\rho,\sigma}(z,x,y) \\
&\quad + \mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x) + \mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y) + \mathfrak{G}_{\sigma,\rho,\kappa}(y,x,z), \\
\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) &= \lim_{t_{\rm src}\to-\infty, t_{\rm snk}\to\infty} e^{m_\mu (t_{\rm snk}-t_{\rm src})} \int_{\alpha,\beta,\eta} G(x,\alpha) G(y,\beta) G(z,\eta) \\
&\quad \times \int_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} S_\mu(x_{\rm snk},\beta)\, i\gamma_\sigma S_\mu(\beta,\eta) i\gamma_\kappa S_\mu(\eta,\alpha) i\gamma_\rho S_\mu(\alpha, x_{\rm src}),
\end{aligned}
$$

Subtraction to (1) remove infrared divergence; (2) reduce discretization and finite volume effects.

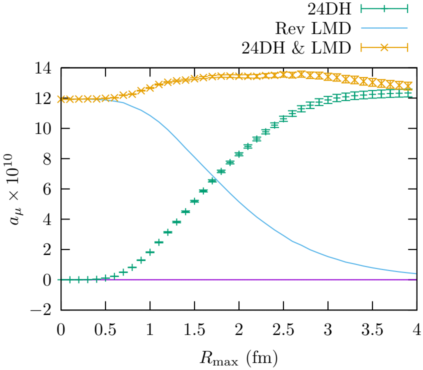
$$
\begin{aligned}
\mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) &= \frac{1}{2}\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \frac{1}{2}[\mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y)]^\dagger, \\
\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,x) &= \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(z,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,z).
\end{aligned}
$$

T. Blum et al 2017. (PRD 96 3, 034515)



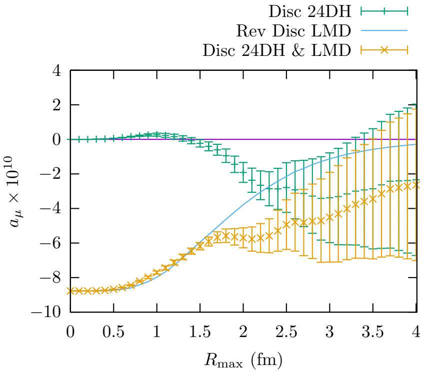
- QED$_L$: $\mathcal{O}(1/L^2)$ finite volume effects
- QED$_\infty$ (no sub) $\mathfrak{G}^{(1)}$: $\mathcal{O}(e^{-mL})$ finite volume effects
- QED$_\infty$ (with sub) $\mathfrak{G}^{(2)}$: smaller $\mathcal{O}(e^{-mL})$ finite volume effects

T. Blum et al 2017. (PRD 96 3, 034515)



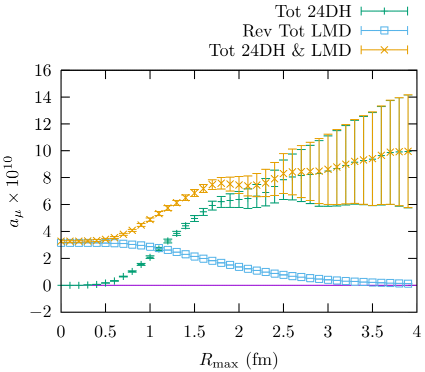
- $a = 0.2$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto $R_{\max}$.
- Rev LMD:
  reverse partial sum down to $R_{\max}$.
- 24DH & LMD:
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times 34/9$.
- At 2.0 fm, the combination gives: $a_\mu^{\text{con}} = 13.44(10)_{\text{stat}} \times 10^{-10}$.



- $a = 0.2$ fm.
- $R_{\rm max} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto $R_{\rm max}$.
- Rev LMD: reverse partial sum down to $R_{\rm max}$.
- 24DH & LMD: the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times(-25/9)$.
- At 2.0 fm, the combination gives: $a_\mu^{\rm discon} = -5.70(58)_{\rm stat} \times 10^{-10}$.

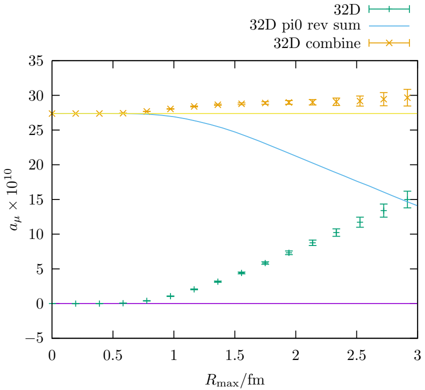


- $a = 0.2$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto $R_{\max}$.
- Rev LMD:

  reverse partial sum down to $R_{\max}$.
- 24DH & LMD:

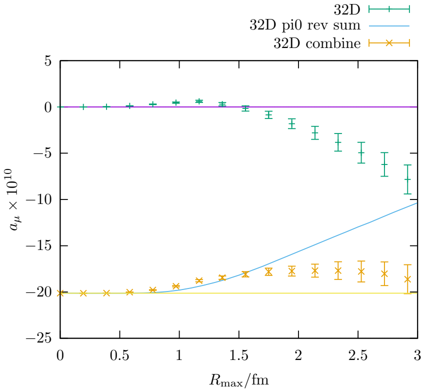
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by the LMD model.
- At 2.0 fm, the combination gives: $a_\mu^{\text{tot}} = 7.46(62)_{\text{stat}} \times 10^{-10}$.



- $a = 0.2$ fm.
- $R_{\text{max}} = \max(|x-y|, |x-z|, |y-z|)$.
- 32D: partial sum upto R_{max}.
- Rev LMD:
  reverse partial sum down to R_{max}.
- 32D & LMD:
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times 34/9$.
- At 2.5 fm, the combination gives: $a_\mu^{\text{con}} = 29.19(73)_{\text{stat}} \times 10^{-10}$.

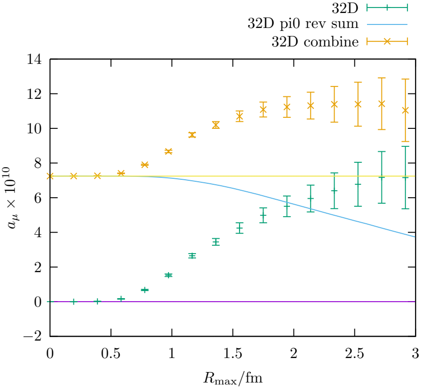


- $a = 0.2$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 32D: partial sum upto $R_{\max}$.
- Rev LMD:

  reverse partial sum down to $R_{\max}$.
- 32D & LMD:

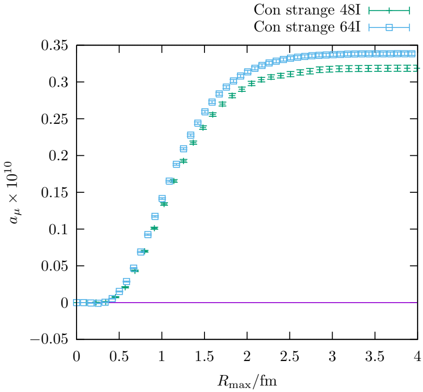
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times(-25/9)$.
- At 2.5 fm, the combination gives: $a_\mu^{\text{discon}} = -17.79(58)_{\text{stat}} \times 10^{-10}$.



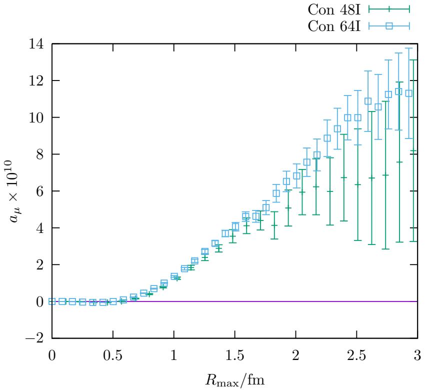
- $a = 0.2$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 32D: partial sum upto $R_{\max}$.
- Rev LMD:
  reverse partial sum down to $R_{\max}$.
- 32D & LMD:
  the sum of the above two curves.

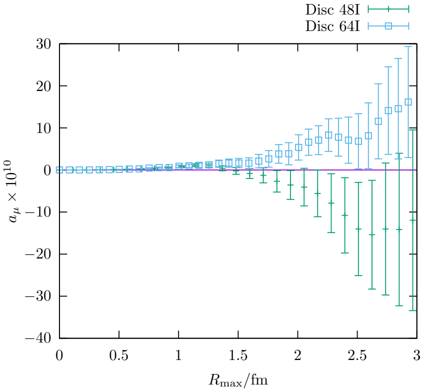
- SD from lattice data. LD part from the LMD model.
- At 2.5 fm, the combination gives: $a_\mu^{\text{tot}} = 11.40(1.27)_{\text{stat}} \times 10^{-10}$.
- Need smaller lattice spacing to control the discretization effects.



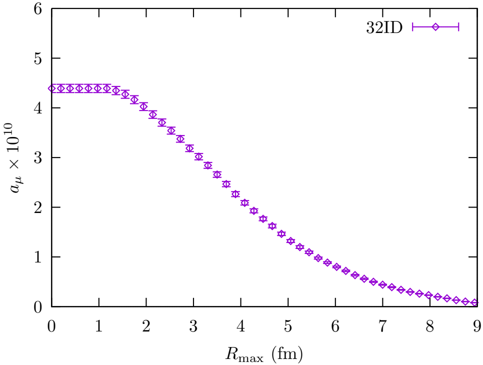
- 48I: $a_\mu^{\text{con-strange}} = 0.319(5)_{\text{stat}} \times 10^{-10}$.
- 64I: $a_\mu^{\text{con-strange}} = 0.338(3)_{\text{stat}} \times 10^{-10}$.
- Continuum limit: $a_\mu^{\text{con-strange}} = 0.361(7)_{\text{stat}} \times 10^{-10}$.

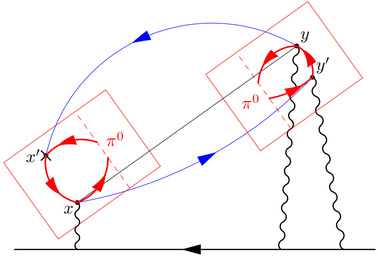
- 48I: $a = 0.114$ fm. 64I: $a = 0.084$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum upto $R_{\max}$.





- 48I: $a = 0.114$ fm. 64I: $a = 0.084$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum upto $R_{\max}$.
- Plan to add more statistics for the 48I ensemble.

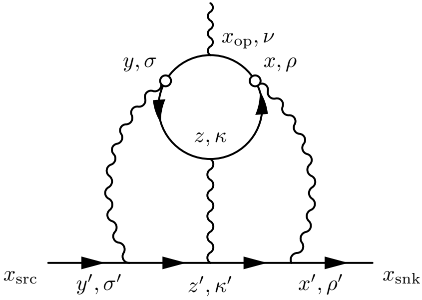




- 32ID: $32^3 \times 64$, $a^{-1} = 1.015$ GeV, $M_\pi = 142$ MeV.
- $R_{\max} = \max(|x-y|, |x-y'|, |y-y'|)$. Reverse partial sum plotted.
- Not the same as the dispersive pion-pole contribution.

Calculation by UCONN graduate student Cheng

Thank You!



- Two point sources at x, y: randomly sample x and y.
- Importance sampling: focus on small $|x - y|$.
- Complete sampling for $|x - y| \leq 5a$ upto discrete symmetry.

$$\frac{a_\mu}{m_\mu} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_{r=x-y} \sum_{z} \sum_{x_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \bar{u}_{s'}(\vec{0}) i \vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\rm op}) u_s(\vec{0})$$

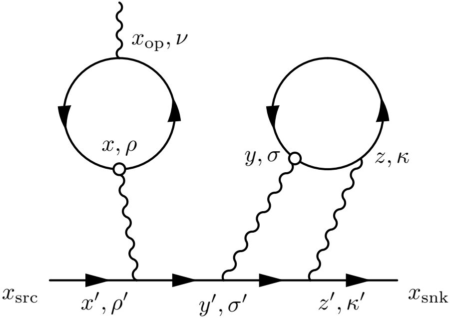
$$\vec{\mu} = \sum_{\vec{x}_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \vec{J}(\vec{x}_{\rm op})$$

Reorder summation

$$|x - y| \leq \min(|y - z|, |x - z|)$$

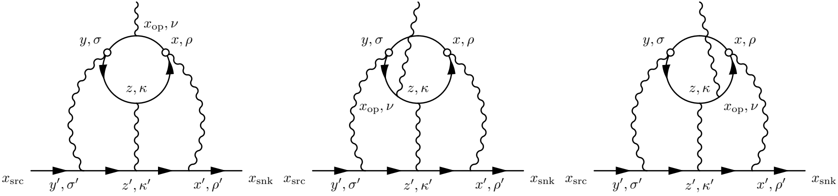
- Muon is plane wave, $x_{\rm ref} = (x + y)/2$.
- Sum over time component for $x_{\rm op}$.
- Only sum over $r = x - y$.

T. Blum et al 2016. (PRD 93, 1, 014503)



- Point x is used as the reference point for the moment method.
- We can use two point source photons at x and y, which are chosen randomly. The points x_{op} and z are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = x - y$.

T. Blum et al 2017. (PRL 118, 2, 022005)



- The three internal vertex attached to the quark loop are equivalent (all permutations are included).
- We can pick the closer two points as the point sources x, y.

$$\sum_{x,y,z} \to \sum_{x,y,z} \begin{cases} 3 & \text{if } |x-y| < |x-z| \text{ and } |x-y| < |y-z| \\ 3/2 & \text{if } |x-y| = |x-z| < |y-z| \\ 3/2 & \text{if } |x-y| = |y-z| < |x-z| \\ 1 & \text{if } |x-y| = |y-z| = |x-z| \\ 0 & \text{others} \end{cases}$$

Split the a_μ^{con} into two parts:

$$a_\mu^{\text{con}} = a_\mu^{\text{con,short}} + a_\mu^{\text{con,long}}$$

- $a_\mu^{\text{con,short}} = a_\mu^{\text{con}}(r \leq 1\text{fm})$:
  most of the contribution, small statistical error.

- $a_\mu^{\text{con,long}} = a_\mu^{\text{con}}(r > 1\text{fm})$:
  small contribution, large statistical error.

Perform continuum extrapolation for short and long parts separately.

- $a_\mu^{\text{con,short}}$: conventional a^2 fitting.

- $a_\mu^{\text{con,long}}$: simply use 48I value.
  Conservatively estimate the relative $\mathcal{O}(a^2)$ error: it may be as large as for $a_\mu^{\text{con,short}}$ from 48I.

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(1/L^3)$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^3} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^2 \log(a^2))$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Bigg(1 - \frac{b_2}{(m_\mu L)^2} - \Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big) \times\Big(1 - \frac{\alpha_S}{\pi}\log\big((a\ \mathrm{GeV})^2\big)\Big)\Bigg)$$

$$a_\mu(L, a^{\text{I}}, a^{\text{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\text{I}}(a^{\text{I}}\ \text{GeV})^2 - c_1^{\text{D}}(a^{\text{D}}\ \text{GeV})^2 + c_2^{\text{D}}(a^{\text{D}}\ \text{GeV})^4\Big)$$

$\mathcal{O}(a^4)$ (maximum of the following two)

$$a_\mu(L, a^{\text{I}}, a^{\text{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\text{I}}(a^{\text{I}}\ \text{GeV})^2 - c_1^{\text{D}}(a^{\text{D}}\ \text{GeV})^2 + c_2(a\ \text{GeV})^4\Big)$$

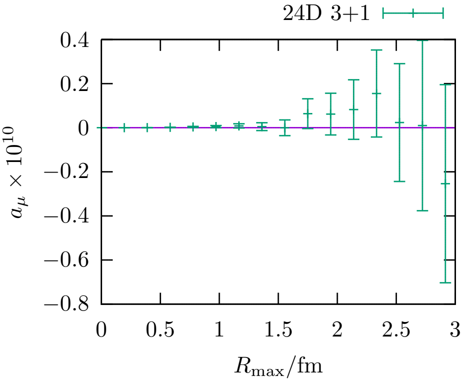
$$a_\mu(L, a^{\text{I}}, a^{\text{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1(a\ \text{GeV})^2 + c_2^{\text{I}}(a^{\text{I}}\ \text{GeV})^4 + c_2^{\text{D}}(a^{\text{D}}\ \text{GeV})^4\Big)$$

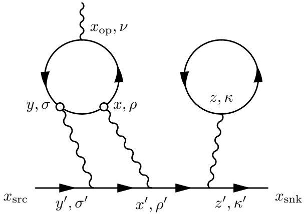
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^2/L)$ (maximum of the following two)

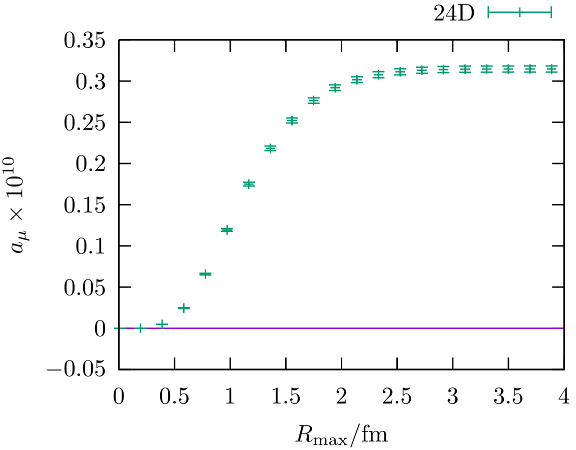
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - \Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)\Big(1 - \frac{1}{m_\mu L}\Big)\right)$$

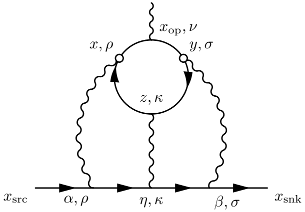
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2}\Big) \times \Big(1 - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$





- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV
- Partial sum upto $R_{\rm max}$
  $R_{\rm max} = \max(|x - y|, |x - z|, |y - z|)$
- The tadpole part comes from C. Lehner et al. 2016 (PRL 116, 232002)
- Systematic error (subdiscon): 0.5×10^{-10}

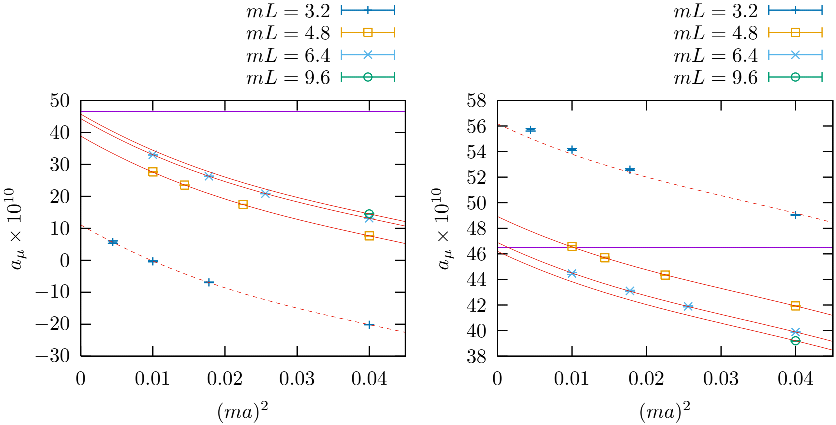




- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum upto $R_{\rm max}$
  $R_{\rm max} = \max(|x-y|, |x-z|, |y-z|)$
- Systematic error (strange con): 0.3×10^{-10}

- Compare the two $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$ in **pure QED computation**.

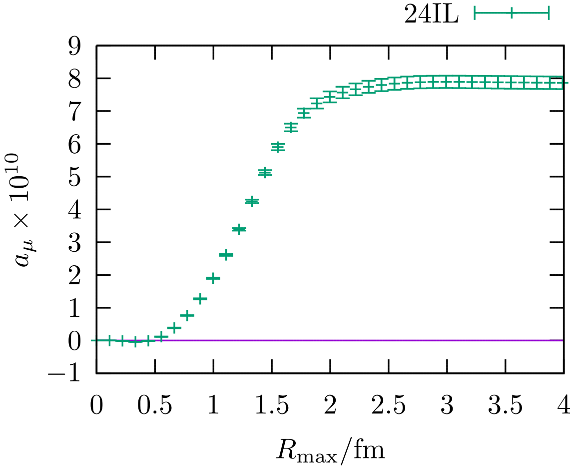


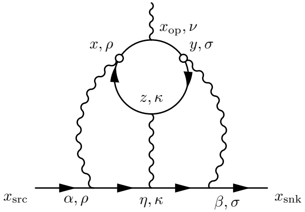
Original QED kernel

Subtracted QED kernel

- Notice the vertical scales in the two plots are different.

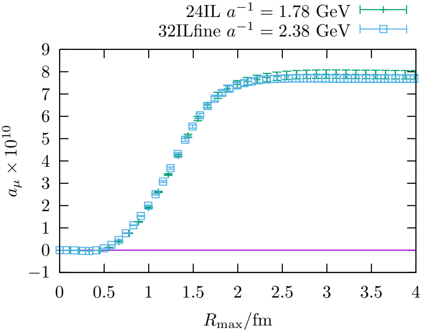
T. Blum et al 2017. (PRD 96 3, 034515)

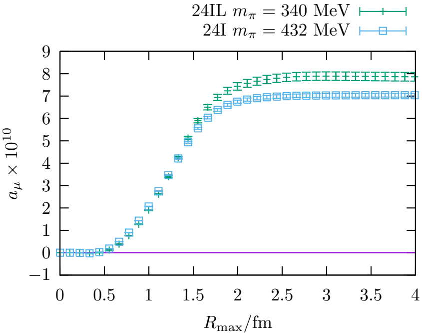




- 24IL: $24^3 \times 64$
  $L = 2.66$ fm
- $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 594$ MeV

- Partial sum upto $R_{\max}$
  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$

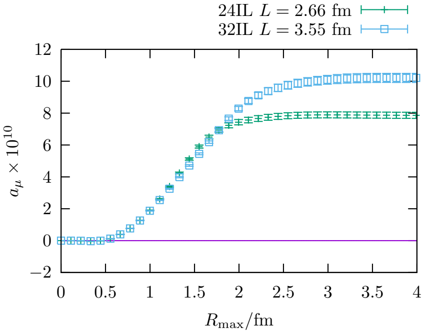


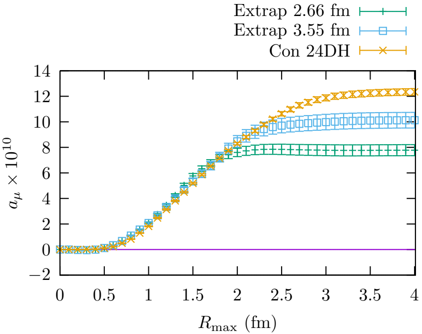


- 24IL: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 32ILfine: $32^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 2.38$ GeV
  $M_\pi = 357$ MeV
  $M_K = 590$ MeV
- 24I: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 432$ MeV
  $M_K = 626$ MeV

$$a_\mu(m_\pi, a, L) = a_\mu(m_\pi^{\text{target}}, 0, L) + c_1 a^2 + c_2(m_\pi^2 - (m_\pi^{\text{target}})^2) \quad (1)$$

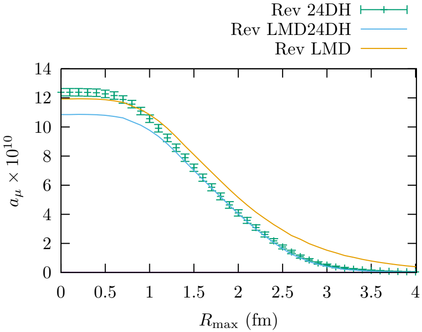
HLbL QED$_\infty$: Inf volume @ $m_\pi = 340$ MeV (prelim)

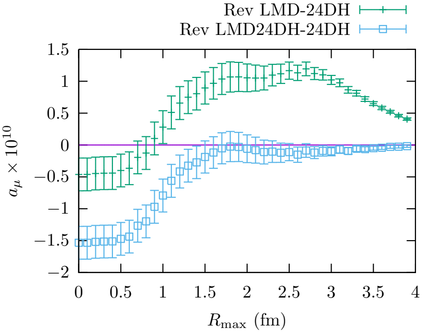




- 24IL: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 32IL: $32^3 \times 64$
  $L = 3.55$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 24DH: $24^3 \times 64$
  $L = 4.67$ fm
  $a^{-1} = 1.015$ GeV
  $M_\pi = 340$ MeV
  $M_K \approx 593$ MeV

$$a_\mu(m_\pi, a, L) = a_\mu(m_\pi^{\text{target}}, 0, L) + c_1 a^2 + c_2(m_\pi^2 - (m_\pi^{\text{target}})^2) \quad (2)$$



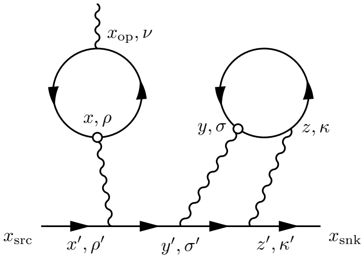


- Reverse partial sum down to $R_{\text{max}} = \max(|x-y|, |x-z|, |y-z|)$
- LMD: Lowest Meson Dominance Model.

  Pion-pole contribution calculated in position space.

  At $m_\pi = 340$ MeV: $f_\pi = 149$ MeV, $M_V = 830$ MeV

  The pion pole contribution should be multiplied by $-34/9$ to match with connected diagram.



- For QED_L, we can compute the QED function for all x given the y location fixed and z summed over. Allow us to compute all combination of x, y with little cost.
- For QED_∞, although we can compute all the function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing y) for all x and z, simply because of its cost is proportion to Volume^2.
- However, we with QED_∞ and interpolation, we can freely choose which coordinates we compute. For example, we may compute all z for $|z - y| \leqslant 5$, and sample z for $|z - y| > 5$.