The hadronic light-by-light contribution to $(g-2)_\mu$: status and introduction

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FOR FUNDAMENTAL PHYSICS

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in memoriam Simon Eidelman

Outline

Introduction: the HLbL contribution to $(g-2)_\mu$

Dispersive approach to the hadronic light-by-light tensor

Short-distance constraints

Conclusions

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White Paper (2020): $(g-2)_\mu$, experiment vs SM

Contribution	Value $\times 10^{11}$
HVP LO (e^+e^-)	6931(40)
HVP NLO (e^+e^-)	−98.3(7)
HVP NNLO (e^+e^-)	12.4(1)
HVP LO (lattice, *udsc*)	7116(184)
HLbL (phenomenology)	92(19)
HLbL NLO (phenomenology)	2(1)
HLbL (lattice, *uds*)	79(35)
HLbL (phenomenology + lattice)	90(17)
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 061(41)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)

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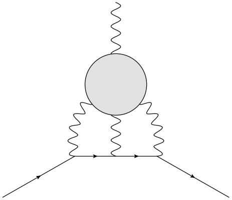
Calculating the HLbL contribution

The HLbL contribution is a very complex quantity

- 4-point function of em currents in QCD



- a data-driven approach, like for HVP, has only recently been developed and used

GC, Hoferichter, Procura, Stoffer=CHPS (14,15,17), Hoferichter, Hoid, Kubis, Leupold, Schneider (18)

- lattice QCD is becoming competitive

Friday session, RBC/UKQCD (20), Mainz (21)

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer ≡ CHPS (2015)

- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the HLbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

HLbL contribution: Master Formula

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

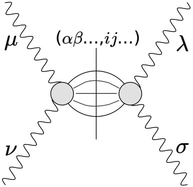
Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$. CHPS (15)

- T_i: known kernel functions
- $\bar{\Pi}_i$ are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses

"Amenable to a dispersive treatment"



+ crossed diagrams

$$\operatorname{Im}\Pi^{\mu\nu\lambda\sigma} = \sum_{\alpha\beta\ldots,ij\ldots} \Gamma^{\mu\nu\alpha\beta\ldots}_{ij\ldots}\Gamma^{\lambda\sigma\alpha\beta\ldots\,\star}_{ij\ldots}$$

- projection on the BTT basis for $\Pi^{\mu\nu\lambda\sigma}$ $\Rightarrow$ DR for Π_i
- result for $\Pi^{\mu\nu\lambda\sigma}$ (and a_μ) depends on the basis choice unless a set of sum rules is satisfied CHPS 17
- even for single-particle intermediate states this is in general not the case, other than for pseudoscalars

→ talk by P. Stoffer

Improvements obtained with the dispersive approach

Contribution	PdRV(09) *Glasgow consensus*	N/JN(09)	J(17)	WP(20)
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3) (scalars + tensors)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

- significant reduction of uncertainties in the first three rows

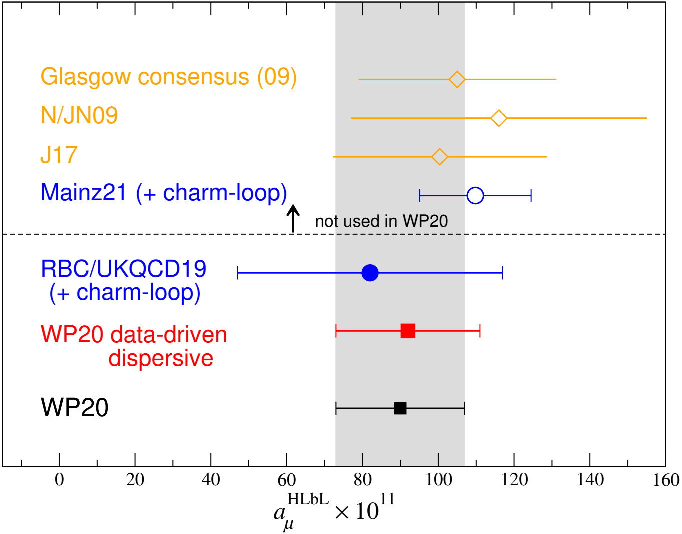
  CHPS (17), Masjuan, Sánchez-Puertas (17) Hoferichter, Hoid et al. (18), Gerardin, Meyer, Nyffeler (19)

  → talk by B. Kubis

- 1 − 2 GeV resonances affected by basis ambiguity → talk by P. Stoffer

- asymptotic region recently addressed, but still work in progress Melnikov, Vainshtein (04), Nyffeler (09), WP rest of this talk, → J. Bijnens and A. Rebhan

Situation for HLbL



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Longitudinal SDCs: a few definitions

The longitudinal SDC only concerns one function: Π_1

Split π^0-pole from the rest in general kinematics ($q_4^2 = 0,\ q_4^\mu \neq 0$):

$$\Pi_1(s,t,u) = \frac{F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi\gamma\gamma^*}(q_3^2)}{s - M_\pi^2} + G(s,t,u)$$

For $g-2$ kinematics ($q_4^\mu \to 0, \Rightarrow\ s = q_3^2,\ t = q_2^2,\ u = q_1^2$):

$$\bar{\Pi}_1(q_3^2,q_2^2,q_1^2) = \frac{F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi\gamma\gamma^*}(q_3^2)}{q_3^2 - M_\pi^2} + G(q_3^2,q_2^2,q_1^2)$$

$$= \frac{F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2)}{q_3^2 - M_\pi^2}\left[F_{\pi\gamma\gamma^*}(M_\pi^2) + \bar{F}_{\pi\gamma\gamma^*}(q_3^2)\right] + G(q_3^2,q_2^2,q_1^2)$$

with $\bar{F}_{\pi\gamma\gamma^*}(q_3^2) \equiv F_{\pi\gamma\gamma^*}(q_3^2) - F_{\pi\gamma\gamma^*}(M_\pi^2)$

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The longitudinal SDCs

Two different kinematic configurations for large q_i^2:

1. All momenta large — Melnikov-Vainshtein (04), Bijnens et al (19)

$$\bar{\Pi}_1(q^2,q^2,q^2) \overset{q^2\to\infty}{=} -\frac{4}{9\pi^2 q^4} + \mathcal{O}(q^{-6})$$

2. $q^2 \equiv q_1^2 \sim q_2^2 \gg q_3^2,\, q^2 \gg \Lambda^2_{\mathrm{QCD}}$: — Melnikov-Vainshtein (04)

$$\bar{\Pi}_1(q_3^2,q^2,q^2) \overset{q^2\to\infty}{=} -\frac{1}{9\pi^2 q^2} w_L(q_3^2) + \mathcal{O}(q^{-4})$$

with $w_L(q_3^2)$ the longitudinal amplitude in $\langle VVA\rangle$, *the anomaly*

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$$\bar{\Pi}_1(q_3^2, q^2, q^2) \overset{q^2 \to \infty}{=} -\frac{1}{9\pi^2 q^2} \frac{6}{q_3^2} + \mathcal{O}(q^{-4})$$

In the chiral (and large-N_c) limit $w_L(q_3^2)$ is known exactly

$$w_L(q_3^2) = \frac{6}{q_3^2} \;\Rightarrow\; G(q_3^2, q^2, q^2)\Big|_{m_q=0} \overset{q \to \infty}{=} \frac{2F_\pi}{3q^2} \frac{\bar{F}_{\pi\gamma\gamma^*}(q_3^2)}{q_3^2}\Big|_{m_q=0} + \mathcal{O}(q^{-4})$$

No individual dispersive contribution satisfies these constraints

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The π-pole for $g-2$ kinematics does — Melnikov-Vainshtein (04)

Recent activity on SDCs (mainly post WP)

- calculation of (non-)perturbative corrections to the OPE

  Bijnens, Hermansson-Truedsson, Laub, Rodríguez-Sánchez (20,21)

- tower of excited pseudoscalars (Regge model)

  GC, Hagelstein, Hoferichter, Laub, Stoffer (19)

- tower of axial-vectors (holographic QCD model)

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Melnikov-Vainshtein and holographic QCD

- Melnikov-Vainshtein model: Melnikov-Vainshtein (04)

$$w_L^{\mathrm{MV}}(q_3^2) = \frac{6}{q_3^2 - M_\pi^2} + \mathcal{O}(M_\pi^2)$$

$$G^{\mathrm{MV}}(q_i^2) = -\frac{F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)\bar{F}_{\pi\gamma\gamma^*}(q_3^2)}{q_3^2} + \mathcal{O}(M_\pi^2)$$

- hQCD (HW2) model: Leutgeb, Rebhan (19), Cappiello et al. (20)

$$w_L^{\mathrm{HW2}}(q_3^2) = \frac{6}{q_3^2 - M_\pi^2}\left[1 + \frac{M_\pi^2 \bar{F}_{\pi\gamma\gamma^*}(q_3^2)}{q_3^2 F_{\pi\gamma\gamma}}\right]$$

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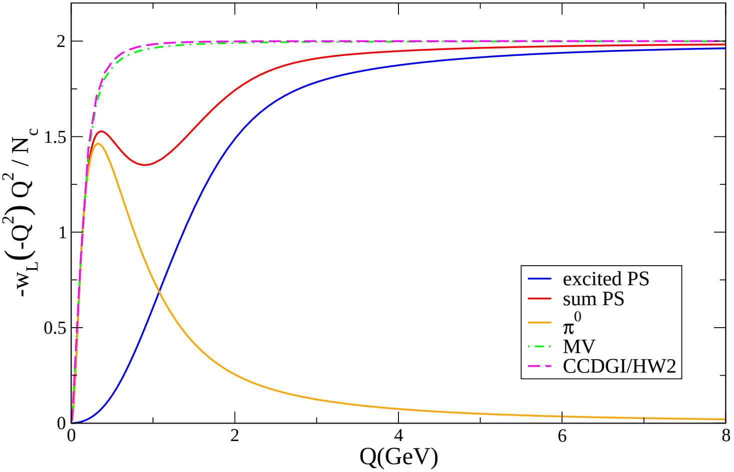
$$\equiv \quad MV(q_i^2) \quad + \quad NF(q_i^2)$$

Numerical comparison for w_L

GC, Hagelstein, Hoferichter, Laub, Stoffer (21)



Numerical comparison for G

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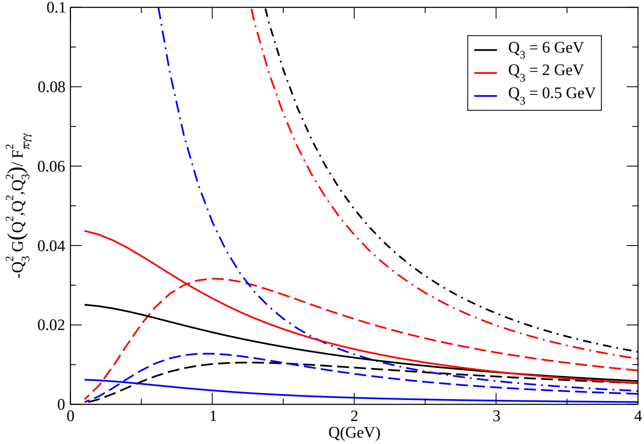

Legenda: dashed=CCDGI/HW2, dotdashed=MV, solid=PS Regge

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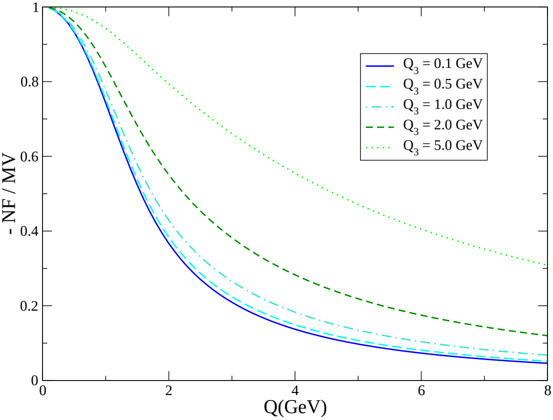

Numerical comparison for a_μ^{HLbL}

GC, Hagelstein, Hoferichter, Laub, Stoffer (21)

	MV model	CCDGI set 1	CCDGI set 2	LR HW2	LR $\text{HW2}_{\text{UV-fit}}$	PS Regge model
			$\Delta a_\mu^{\pi/a_1} \times 10^{11}$			
$Q_i^2 > Q^2_{\text{match}} \quad \forall i$	1.4	0.5	0.8	0.6	0.8	0.7
$Q_{1,2}^2 > Q^2_{\text{match}} > Q_3^2$	0.1	0.0	0.1	0.0	0.1	0.1
$Q_{i,3}^2 > Q^2_{\text{match}} > Q_j^2 \quad i \neq j \neq 3$	2.0	1.0	1.2	1.0	1.2	0.7
$Q_i^2 > Q^2_{\text{match}} > Q_{j,k} \quad i \neq j \neq k$	0.8	0.3	0.4	0.3	0.3	0.2
$Q^2_{\text{match}} > Q_i^2 \quad \forall i$	11.8	2.2	1.7	2.3	1.8	1.0
Total	16.2	4.0	4.2	4.2	4.3	2.7
			$\Delta a_\mu^{\eta/f_1+\eta'/f_1'} \times 10^{11}$			
$Q_i^2 > Q^2_{\text{match}} \quad \forall i$	3.4	1.3	1.7	1.7	2.5	3.1
$Q_{1,2}^2 > Q^2_{\text{match}} > Q_3^2$	0.3	0.1	0.2	0.1	0.2	−0.1
$Q_{i,3}^2 > Q^2_{\text{match}} > Q_j^2 \quad i \neq j \neq 3$	3.7	2.5	2.8	3.0	3.7	2.8
$Q_i^2 > Q^2_{\text{match}} > Q_{j,k} \quad i \neq j \neq k$	1.7	0.8	0.9	0.9	0.9	0.9
$Q^2_{\text{match}} > Q_i^2 \quad \forall i$	12.9	5.6	5.1	6.8	5.5	3.1
Total	22.1	10.3	10.7	12.5	12.8	9.9
Grand total ($\pi/a_1 + \eta/f_1 + \eta'/f_1'$)	38.3	14.3	14.9	16.7	17.1	12.6

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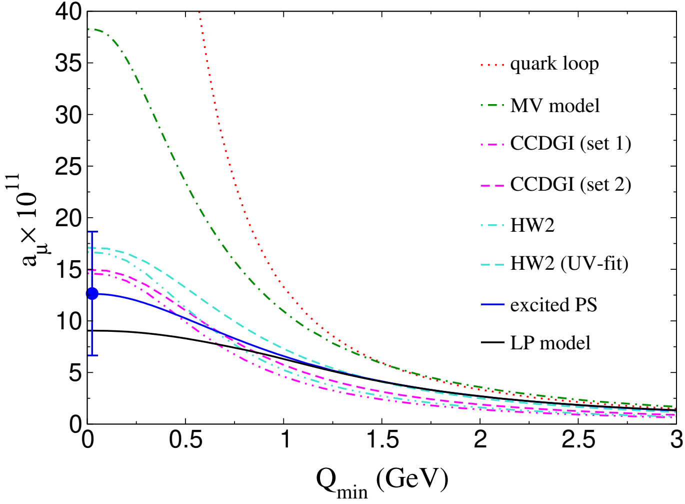

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- the dispersive approach to HLbL has put this contribution on a solid, systematically improvable basis
- there remain conceptual problems to be solved: ambiguities related to the basis choice for the HLbL tensor

  $\longrightarrow$ talk by P. Stoffer
- these affect the contribution of resonances in the narrow-width approximation: scalars, tensors and axials

  $\longrightarrow$ talks by B. Kubis and P. Stoffer
- short-distance constraints are the most important source of uncertainty at present. Recent work has shown that
  - the anomaly plays a minor role for a_μ^{HLbL}
  - the WP estimate is conservative
  - uncertainties can be further reduced (work in progress)

$\longrightarrow$ talks by J. Bijnens and A. Rebhan