SHORT-DISTANCE CONSTRAINTS TO THE MUON $g-2$ HLbL

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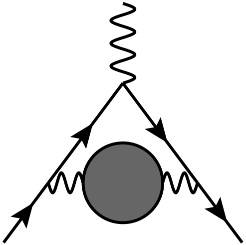
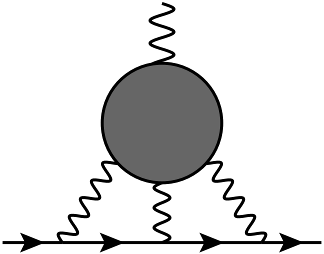
Where are we now: experiment/white paper

- $a_\mu = 116592089(63) \times 10^{-11}$ (BNL)
- $a_\mu = 116592040(54) \times 10^{-11}$ (FNAL)
- $a_\mu = 116592061(41) \times 10^{-11}$ (FNAL+BNL)
- $a_\mu = 116591810(43) \times 10^{-11}$ (White paper)
- $\Delta a_\mu = 251(59) \times 10^{-11}$
- Theory and experiment very similar error: improvement needed on the theory

Hadronic contributions



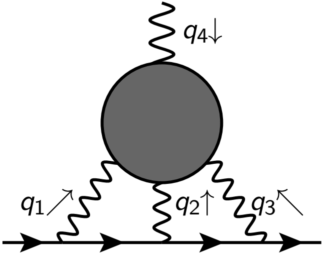
LO-HVP



HLbL

- Muon and photon lines, representative diagrams
- The blobs are hadronic contributions
- Higher order contributions of both types: known accurately enough
- $a_\mu^{HVP} = 6845(40)\ 10^{-11}$ (LO+NLO+NNLO) (see earlier this week)
- $a_\mu^{HLbL} = 92(18)\ 10^{-11}$ (LO+NLO) (today and tomorrow)

HLbL: the main object to calculate

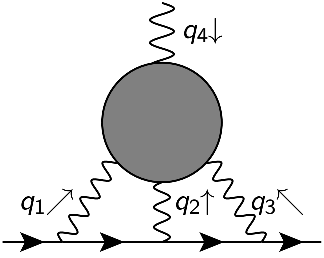


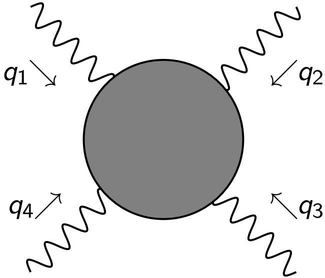
- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- q_4 always at zero
- Double counting needs to be avoided: hadron exchanges versus quarks

Contributions

- Numbers from white paper (updates see talks this afternoon)
- "Long distance": under good control
  - Dispersive method: Berne group around G. Colangelo
  - π^0 (and η, η') pole: $93.8(4.0)\ 10^{-11}$
  - Pion and kaon box (pure): $-16.4(2)\ 10^{-11}$
  - $\pi\pi$-rescattering (include scalars below 1 GeV): $-8(1)\ 10^{-11}$
- Charm (beauty, top) loop: $3(1)\ 10^{-11}$
- "Short and medium distance"
  - Axial vector: $6(6)\ 10^{-11}$
  - Short-distance: $15(10)\ 10^{-11}$
- Clearly the last item needs improvement
- A guesstimate of the overlap went into this

Definitions





$$= \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$$

- Actually we really need $\left.\dfrac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$
- Never purely short-distance: q_4 at zero
- $q_i^2 = -Q_i^2$

Definitions

$$\Pi^{\mu\nu\lambda\sigma} = -i\int d^4x d^4y d^4z e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\left\langle T\left(j^\mu(x)j^\nu(y)j^\lambda(z)j^\sigma(0)\right)\right\rangle$$

Use the Colangelo et al. conventions (mainly)

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\hat{\Pi}_i, \qquad \left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\right|_{q_4=0} = \sum_{i=1}^{54}\left.\frac{\delta T_i^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\hat{\Pi}_i\right|_{q_4=0}$$

$$a_\mu = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3\int_{-1}^1 d\tau\sqrt{1-\tau^2}\sum_{i=1}^{12}\hat{T}_i(Q_1,Q_2,\tau)\overline{\Pi}_i(Q_1,Q_2,\tau)$$

$$Q_3^2 = Q_1^2+Q_2^2+2Q_1Q_2\tau$$

- The 12 $\overline{\Pi}_i$ from $\hat{\Pi}_i$ for $i=1,4,7,17,39,54$
- The integral can be parametrized in many ways

Short-distance constraints

- There are very many different types of short-distance constraints (SDC)
- Those on hadronic properties
  - Couplings of hadrons to off-shell photons
  - Pure OPE (e.g. $\pi^0 \to \gamma^*\gamma^*$ at $Q_1^2 = Q_2^2$)
  - Brodsky-Lepage-Radyushkin-$\cdots$:
    - the overall power is very well predicted (counting rules)
    - the coefficient follows from the asymptotic wave functions and possible α_S corrections: larger uncertainty
  - Light-cone QCD sum rules
  - $\cdots$
- On the full four-point function (4, 3 or 2 currents close)
  - SD4: $\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$ all $Q_i \cdot Q_j$ large
  - SD3: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ with $Q_1^2 \sim Q_2^2 \sim Q_3^2 \gg \Lambda_{QCD}^2$ JB,LL,NHT,ARS 19-21
  - SD2: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ and $Q_1^2 \sim Q_2^2 \gg Q_3^2 (\gg \Lambda_{QCD}^2)$ Melnikov-Vainshtein 03
  - $\cdots$

Implementation

- For hadronic sub-parts: talks by Kubis, Stoffer
- For the full four-point function: talk by Rebhan
- Some general comments:
  - Brodsky-Lepage constraints together with full n-point functions SDC often require an infinite number of resonances for obeying both JB,Gamiz,Lipartia,Prades 2003
  - Have a model that fully implements SDC and then integrate everywhere
  - Have a good description in the intermediate domain, use QCD expressions to do the short-distance part of the integration
  - Varying the transitions can help with error estimates
  - Make sure to avoid double counting: splitting the integration over different regions is a good way to avoid this

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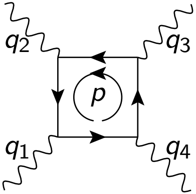
Quark-loop

- Use (constituent) quark-loop
- Used for full estimates since the beginning (1970s)
- Used for short-distance estimates with mass as a cut-off
  JB, Pallante, Prades, 1996
- Has been recalculated by many people in many ways



Quark-loop: u, d, s

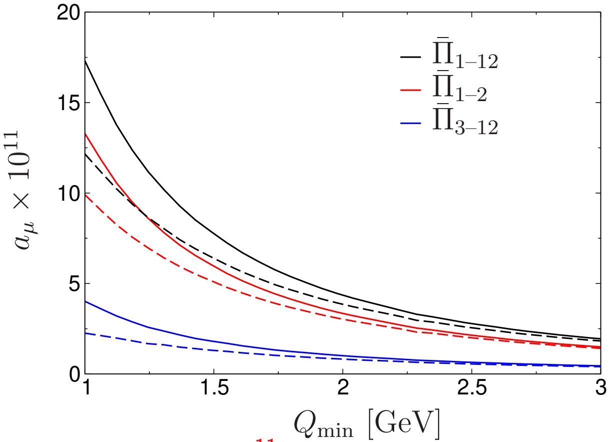


figure: Hoferichter

$Q_1, Q_2, Q_3 > Q_{\min}$

$M_Q = 0$: full

$M_Q = 0.3$ GeV dashed

$a_\mu^{\mathrm{HLbLQ}} = 54 \times 10^{-11}$

- About 12×10^{-11} from above 1 GeV for $M_Q = 0.3$ GeV
- About 17×10^{-11} from above 1 GeV for $M_Q = 0$

Quark-loop

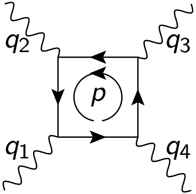
- Is it a first term in a systematic OPE?
- YES: JB, N. Hermansson-Truedsson, L. Laub, A. Rodríguez-Sánchez
  - Phys.Lett. B798 (2019) 134994 [arxiv:1908.03331]: principle and next nonperturbative term
  - JHEP 10 (2020) 203 [arxiv:2008.13487]: proper description and up to NNLO nonperturbative terms
  - JHEP 04 (2021) 240 [arxiv:2101.09169]: perturbative correction
- Higher order terms are not just the quark-loop

Short-distance: first attempt

$$\Pi^{\mu\nu\lambda\sigma} = -i\int d^4x d^4y d^4z e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\left\langle T\left(j^\mu(x)j^\nu(y)j^\lambda(z)j^\sigma(0)\right)\right\rangle$$

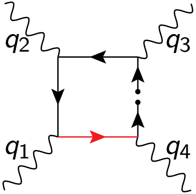
- Usual OPE: x, y, z all small (4 currents close)
- First term in the expansion is the massless quark-loop no problem with $\partial/\partial q_4^\rho$ and $q_4 \to 0$

  p in loop $\Rightarrow$ no singular propagators:



- Next term problems: no loop momentum;

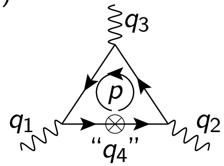
  $q_4 \to 0$ propagator diverges:



- Due to the symmetries: $1/q_4^2$ essentially unavoidable

Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, Balitsky, Yung, 1983
- For the q_4-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  whole calculation is immediately with $q_4 = 0$.
- First term is exactly the massless quark-loop
  (quark masses: next order)



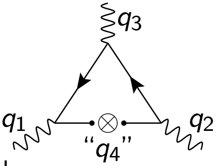
- 3 quark currents close

Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates
- There are new condensates, induced by the constant magnetic field: $\langle \bar{q}\sigma_{\alpha\beta}q\rangle \equiv e_q F_{\alpha\beta} X_q$
- Lattice QCD Bali et al., arXiv:2004.08778
  $X_u = 40.7 \pm 1.3$ MeV,
- Only starts at $1/Q^2$ via $m_q X_q$ corrections to the leading quark-loop result
- X_q and m_q are very small, only a very small correction
- X_q: contain IR divergent perturbative parts, combine with the m_q^2 corrections from the quark-loop consistently
- Next order: very many condensates contribute, lots of IR mixing and redefinitions.
- Infrared divergences absorbed in the condensates

Results

- Result derived from:



- $N_c = 3$ and one quark

$$\hat{\Pi}_1 = m_q X_q e_q^4 \frac{-4(Q_1^2 + Q_2^2 - Q_3^2)}{Q_1^2 Q_2^2 Q_3^4} \qquad \hat{\Pi}_7 = 0$$

$$\hat{\Pi}_4 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2} \qquad \hat{\Pi}_{17} = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^4}$$

$$\hat{\Pi}_{54} = m_q X_q e_q^4 \frac{-4(Q_1^2 - Q_2^2)}{Q_1^4 Q_2^4 Q_3^2} \qquad \hat{\Pi}_{39} = 0$$

Short-distance: nonperturbative numerical results

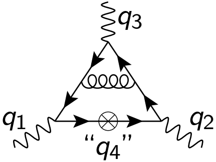
Order	Contribution	$Q_{\min} = 1\,\mathrm{GeV}$	$Q_{\min} = 2\,\mathrm{GeV}$
$1/Q^2_{\min}$	quark-loop	$1.73 \cdot 10^{-10}$	$4.35 \cdot 10^{-11}$
$1/Q^4_{\min}$	quark-loop, m_q^2	$-5.7 \cdot 10^{-14}$	$-3.6 \cdot 10^{-15}$
	$X_{2,m}$	$-1.2 \cdot 10^{-12}$	$-7.3 \cdot 10^{-14}$
$1/Q^6_{\min}$	X_{2,m^3}	$6.4 \cdot 10^{-15}$	$1.0 \cdot 10^{-16}$
	X_3	$-3.0 \cdot 10^{-14}$	$-4.7 \cdot 10^{-16}$
	X_4	$3.3 \cdot 10^{-14}$	$5.3 \cdot 10^{-16}$
	X_5	$-1.8 \cdot 10^{-13}$	$-2.8 \cdot 10^{-15}$
	X_6	$1.3 \cdot 10^{-13}$	$2.0 \cdot 10^{-15}$
	X_7	$9.2 \cdot 10^{-13}$	$1.5 \cdot 10^{-14}$
	$X_{8,1}$	$3.0 \cdot 10^{-13}$	$4.7 \cdot 10^{-15}$
	$X_{8,2}$	$-1.3 \cdot 10^{-13}$	$-2.0 \cdot 10^{-15}$

- $Q_1, Q_2, Q_3 \geq Q_{\min}$
- Nonperturbative short-distance corrections are small
- Suppression by small quark masses or small condensates
- Nonperturbative short-distance corrections are small

Short-distance: $1/Q^2_{\rm min}$

- Can we understand scaling with $Q_{\rm min}$?
- $a_\mu = \dfrac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2\, Q_1^3 Q_2^3 \int_{-1}^{1} d\tau\sqrt{1-\tau^2}\sum_{i=1,12}\hat{T}_i\overline{\Pi}_i$
- Do $Q_i \to \lambda Q_i$
- overall factor goes as λ^8
- Quark loop has no scale thus $\hat{\Pi}_i$ scale with their dimension
  $\hat{\Pi}_1, \hat{\Pi}_4 \sim \lambda^{-4}, \qquad \hat{\Pi}_7, \hat{\Pi}_{17}, \hat{\Pi}_{39}, \hat{\Pi}_{54} \sim \lambda^{-6}$
- $\Longrightarrow \overline{\Pi}_{1,\dots,4} \sim \lambda^{-4} \qquad \overline{\Pi}_{5,\dots,12} \sim \lambda^{-6}$
- Expand the T_i for $Q_i \gg m_\mu$: $T_1 \sim m_\mu^4$, $T_{i\neq 1} \sim m_\mu^2$
  $T_1 \sim \lambda^{-8}$, $T_{2,3,4} \sim \lambda^{-6}$, $T_{5,\dots,12} \sim \lambda^{-4}$
- Put all together: quark-loop scales as $a_\mu^{\rm SD\ ql} \sim \lambda^{-2}$
- $m_q X_q$ adds an overall factor $\Longrightarrow a_\mu^{{\rm SD}X_q} \sim \lambda^{-4}$
- Note it agrees with Dominik's argument

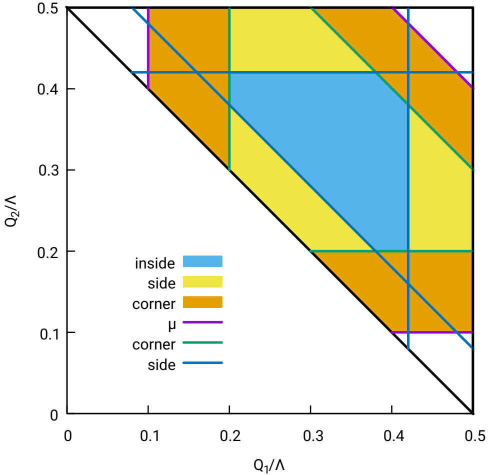
Perturbative corrections

- Representative diagram: 

- All integrals are known
- Infrared and UV divergences in individual diagrams
- Dimensional regularization: $d = 4 - 2\epsilon$
- All $1/\epsilon^3, 1/\epsilon^2, 1/\epsilon$ cancel
- Several independent calculations that agree
- Find some typos in integral papers (I hate signs)

Perturbative corrections

- Use method of master integrals: disadvantage: large numerical cancellations between integrals
- Especially near $\lambda = Q_1^4 + Q_2^4 + Q_3^4 - 2Q_1^2Q_2^2 - 2Q_2^2Q_3^2 - 2Q_3^2Q_1^2 = 0$



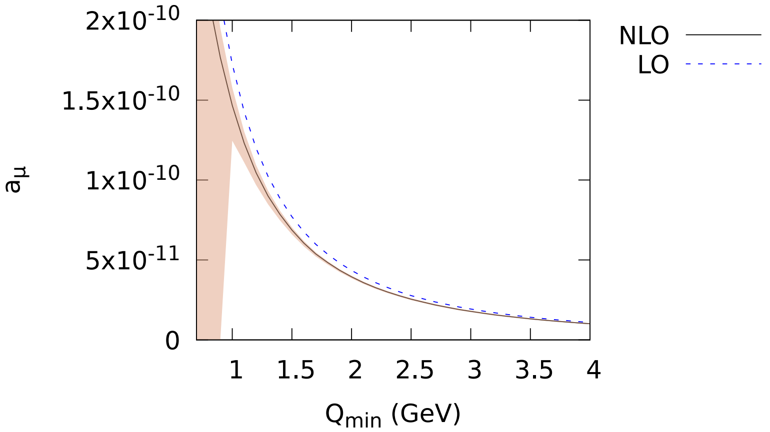
- $Q_1 + Q_2 + Q_3 = \Lambda$
- $Q_1, Q_2, Q_3 \geq \mu = Q_{\text{min}}$
- Need to expand on sides and corners
- Up to $1/\lambda^4$ occurs
- Analytical expressions for all regions available
- Simple for symmetric point and corners

Perturbative corrections: numerics

	Quark loop	Gluon corrections ($\frac{\alpha_s}{\pi}$ units)
$\bar{\Pi}_1$	0.0084	-0.0077
$\bar{\Pi}_2$	13.28	-12.30
$\bar{\Pi}_3$	0.78	-0.87
$\bar{\Pi}_4$	-2.25	0.62
$\bar{\Pi}_5$	0.00	0.20
$\bar{\Pi}_6$	2.34	-1.43
$\bar{\Pi}_7$	-0.097	0.056
$\bar{\Pi}_8$	0.035	0.41
$\bar{\Pi}_9$	0.623	-0.87
$\bar{\Pi}_{10}$	1.72	-1.61
$\bar{\Pi}_{11}$	0.696	-1.04
$\bar{\Pi}_{12}$	0.165	-0.16
Total	17.3	-17.0

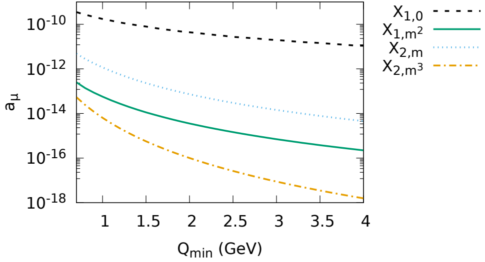
- a_μ from integration from $Q_{\min} = 1\,\mathrm{GeV}$ in 10^{-11} units.
- Naive scaling to other Q_{min} applies (up to $\alpha_S(Q_{\min})$)
- $a_\mu^{\mathrm{HLbL\ SD\ gluonic}} = -1.7\ 10^{-11}$
- $Q_{\min} = 1$ GeV, $\alpha_S = 0.33$
- Main uncertainty: how to handle α_S
- No sign that it is very large (about -10%)

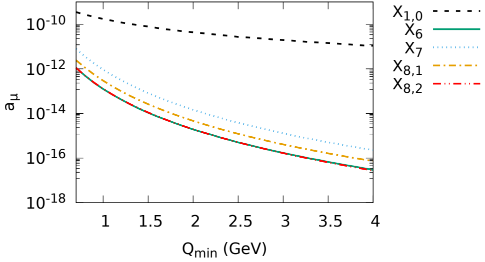
Perturbative corrections: numerics

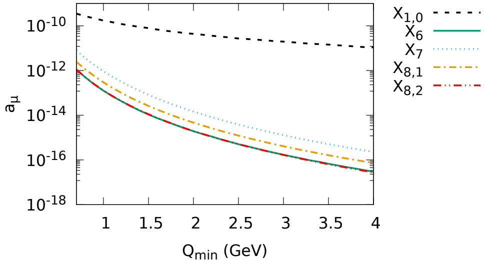


- Uncertainty estimated by $\alpha_S(\mu)$ with $Q_{\text{min}}/\sqrt{2} \leq \mu \leq \sqrt{2}Q_{\text{min}}$
- Running $\alpha_S(M_Z)$ at 5 loops to $\alpha_S(m_\tau)$ or $\alpha_S(\mu)$

Nonperturbative corrections: numerics

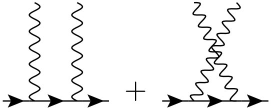






MV short-distance

- K. Melnikov, A. Vainshtein, Phys. Rev. **D70** (2004) 113006. [hep-ph/0312226]
- take $Q_1^2 \approx Q_2^2 \gg Q_3^2$: Leading term in OPE of two vector currents is proportional to axial current
- $\Pi^{\rho\nu\alpha\beta} \propto \frac{P_\rho}{Q_1^2} \langle 0 | T \left(J_{A\nu} J_{V\alpha} J_{V\beta} \right) | 0 \rangle$
- J_A comes from



- Coefficient of J_A has α_S and higher order OPE corrections
- AVV triangle anomaly: in particular nonrenormalization theorems
  - fully for longitudinal ($\overline{\Pi}_i, i = 1, 2, 3$)
  - perturbative for the others
- Implications recent overview: M. Knecht, JHEP 08 (2020) 056 [arXiv:2005.09929], P. Masjuan, P. Roig, and P. Sanchez-Puertas, arXiv:2005.11761
- See also Colangelo et al, JHEP 03 (2020) 101 [arXiv:1910.13432], arXiv:2106.13222

Short-distance: MV

- Only a proper prediction for $\hat{\Pi}_1$
- $\overline{Q}_3 = Q_1 + Q_2$, $Q_3 \ll Q_1, Q_2$
- $\hat{\Pi}_1 = \frac{e_q^4}{\pi^2} \frac{-12}{Q_3^2 \overline{Q}_3^2} \left(1 - \frac{\alpha_S}{\pi}\right)$
- The quark-loop and its gluonic correction reproduce this
- JB,NHT,ARS in progress: calculate the corrections: gluonic and OPE
- Next term in OPE has a number of features (from our corner expansions):
  - $\log \frac{Q_3^2}{\overline{Q}_3^2}$ show up already at $\alpha_S = 0$
  - For some of the terms the gluonic corrections dominate
- Should allow to resum some of the large corrections of the corners

Conclusions

- We have shown that the massless quarkloop really is the first term of a proper OPE expansion for the HLbL
- We have shown how to properly go to higher orders
- We have calculated the next two terms in the OPE
  - NLO: suppressed by quark masses and a small X_q
  - NNLO: large number of induced condensates but all small
  - Numerically not relevant at the present precision
- Gluonic corrections about -10%
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain
- Finding the onset of the asymptotic domain
- The MV limit provides constraints on models (but there are α_S and higher order corrections)