Transition form factors — η and f_1

Bastian Kubis

HISKP (Theorie) & BCTP

Universität Bonn, Germany

Muon $g-2$ theory initiative workshop

KEK, 1/7/2021

Simon Eidelman (1948–2021)

picture credit: Zdeněk Doležal 2014

Outline

Paradigm: π^0 transition form factor

η transition form factor: from singly- to doubly-virtual

- analysis of $e^+e^- \to \eta\pi^+\pi^-$ and comparison to $\eta \to \pi^+\pi^-\gamma$ Simon Holz, Plenter et al., arXiv:1509.02194v2

Axial-vectors

- form factor phenomenology for the $f_1(1285)$ Marvin Zanke, BK, Hoferichter, arXiv:2103.09829

Summary / Outlook

Paradigm case: the π^0 transition form factor

Hoferichter, Hoid, BK, Leupold, Schneider 2018

- double-spectral-function representation for π^0 TFF

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\left[\rho^{\rm disp}+\rho^{\rm eff}+\rho^{\rm asym}\right](x,y)}{(x-q_1^2)(y-q_2^2)}$$

  - ▷ $\rho^{\rm disp}$: leading 2π and 3π singularities
  - ▷ $\rho^{\rm eff}$: effective pole (small), fulfils sum rules for

$$F_{\pi^0\gamma^*\gamma^*}(0,0) \quad \text{and} \quad \lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$$ [Brodsky–Lepage]

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  - ▷ $\rho^{\rm asym}$: pQCD asymptotics above matching scale s_m rewritten from pion wave function

- **here:** only improve $\rho^{\rm disp}$ for η cf. S. Holz, talk at Seattle meeting 2019
- implement asymptotics for axial-vector TFF(s)

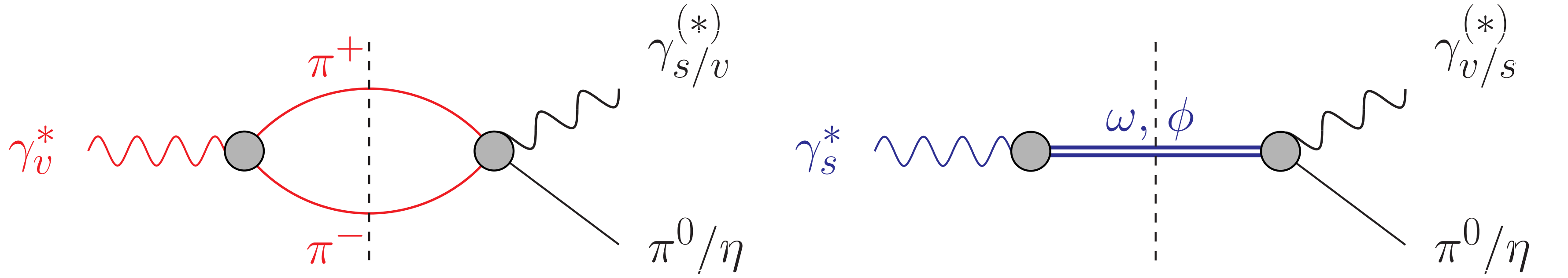
Dispersive analysis of $\pi^0/\eta \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vs}(q_1^2,q_2^2) + F_{vs}(q_2^2,q_1^2)$$
$$F_{\eta\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vv}(q_1^2,q_2^2) + F_{ss}(q_1^2,q_2^2)$$

- leading hadronic intermediate states:

Hanhart et al. 2013, Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor — well known from $e^+e^- \to \pi^+\pi^-$

$\times$ $\eta \to \pi\pi\gamma^*$ P-wave amplitude — Omnès representation

▷ isoscalar photon: 3 pions

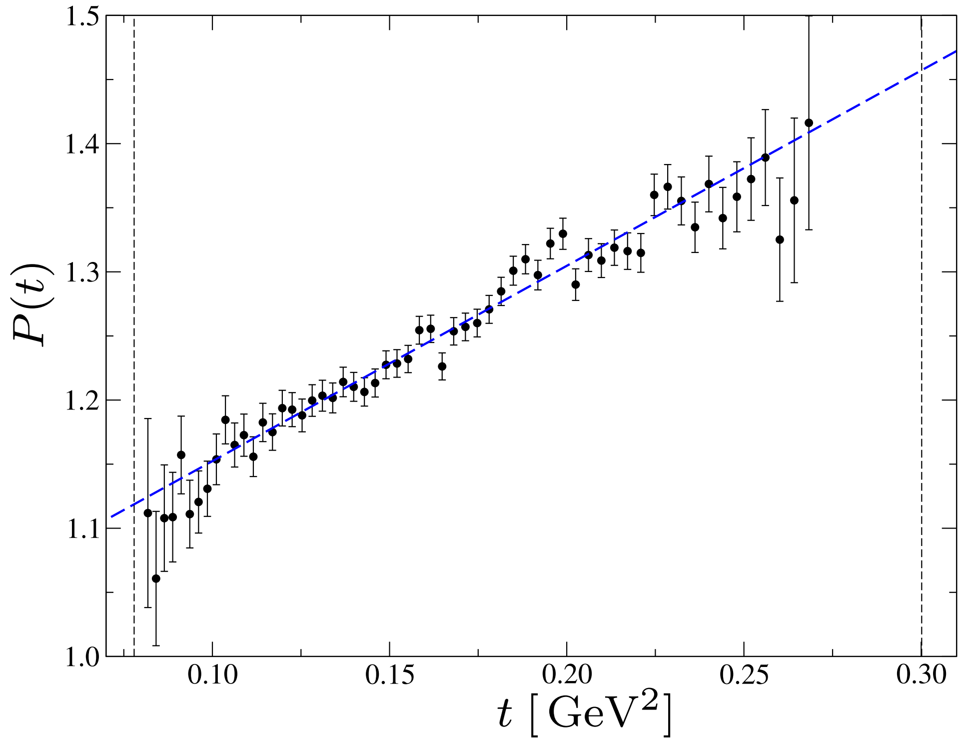
$\longrightarrow$ dominated by narrow ω, ϕ; very small for the η

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

- $\pi^+\pi^-$ in P-wave $\longrightarrow$ universal final-state interactions; ansatz:

$$\mathcal{F}_{\eta^{(\prime)}\pi\pi\gamma}(t) = A \times P(t) \times \Omega(t), \quad P(t) = 1 + \alpha^{(\prime)} t, \quad t = M_{\pi\pi}^2$$

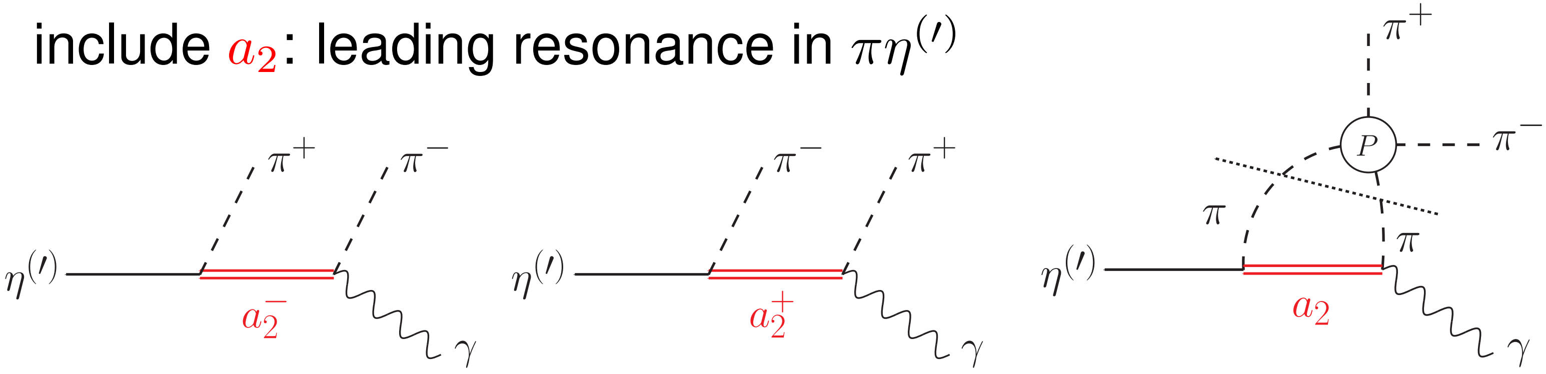
- divide data by Omnès function $\Omega(t) \longrightarrow P(t)$ Stollenwerk et al. 2012

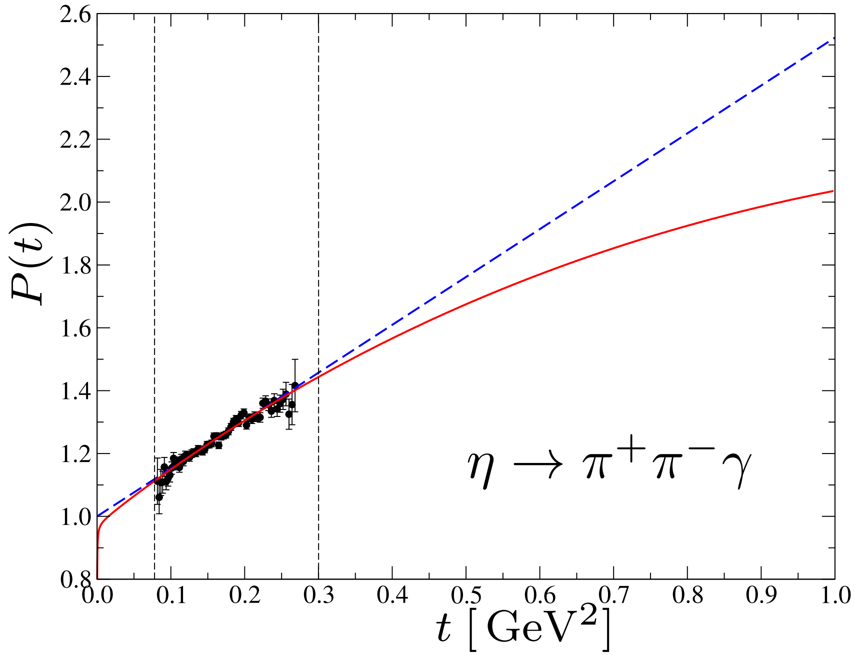


data: KLOE 2013

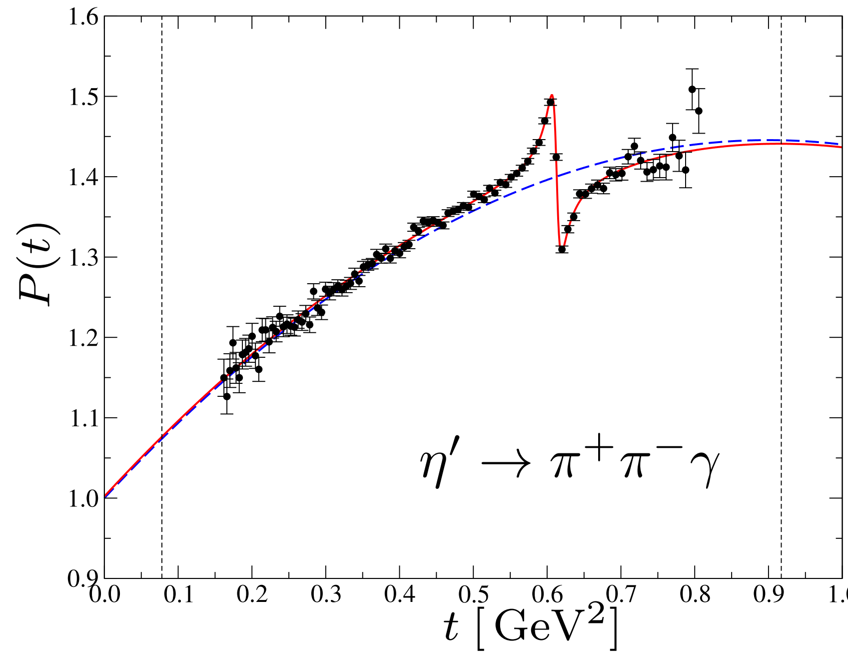
$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$





KLOE 2013; BK, Plenter 2015



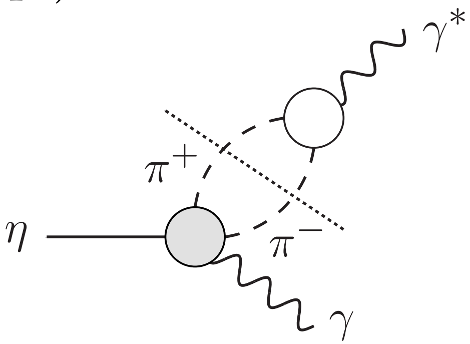
BESIII 2017; Hanhart et al. 2017

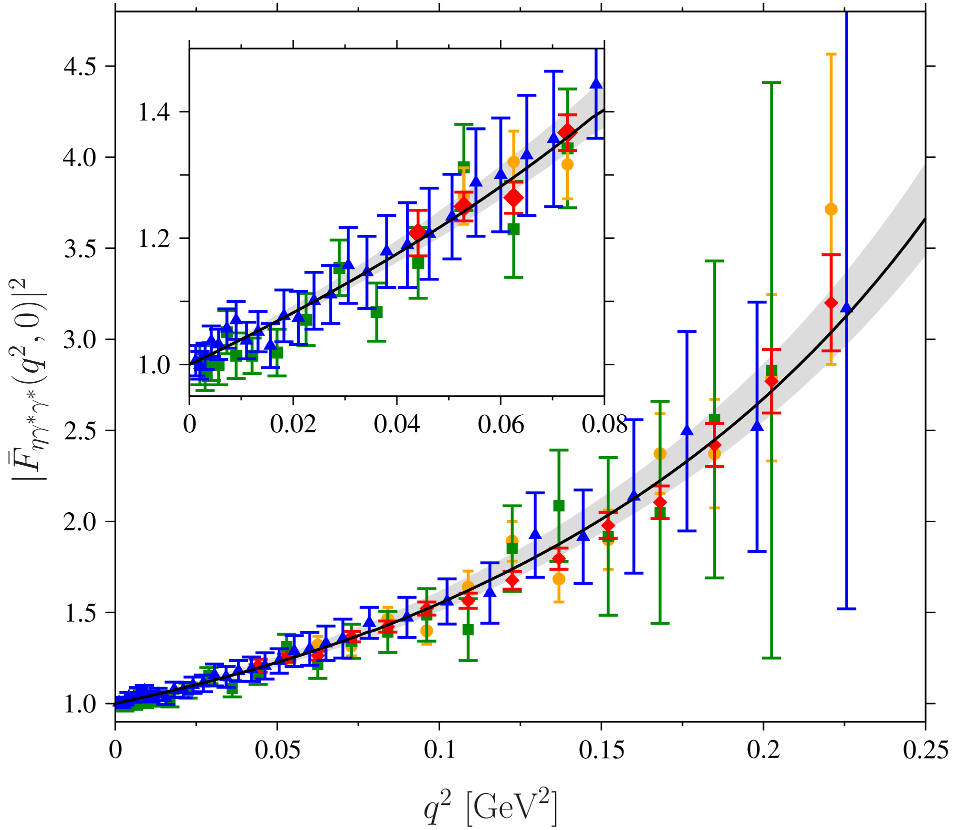
- induces curvature in $P(t)$
- curvature, plus ρ–ω mixing

Transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013, BK, Plenter 2015

$$F_{\eta\gamma^*\gamma}(q^2,0) = F_{\eta\gamma\gamma} + \frac{q^2}{12\pi^2}\int_{4M_\pi^2}^{\infty} dt\, \frac{q_\pi^3(t)[F_\pi^V(t)]^* F_{\eta\pi\pi\gamma}(t)}{t^{3/2}(t-q^2)}$$
$$+ \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0)\ [\longrightarrow \text{VMD}]$$



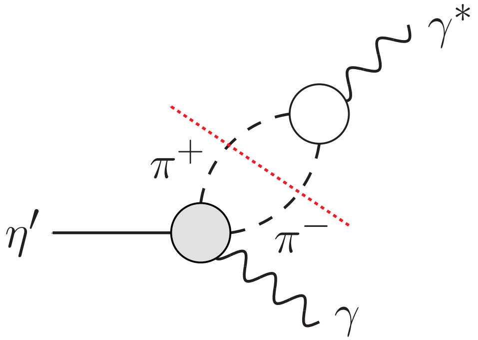


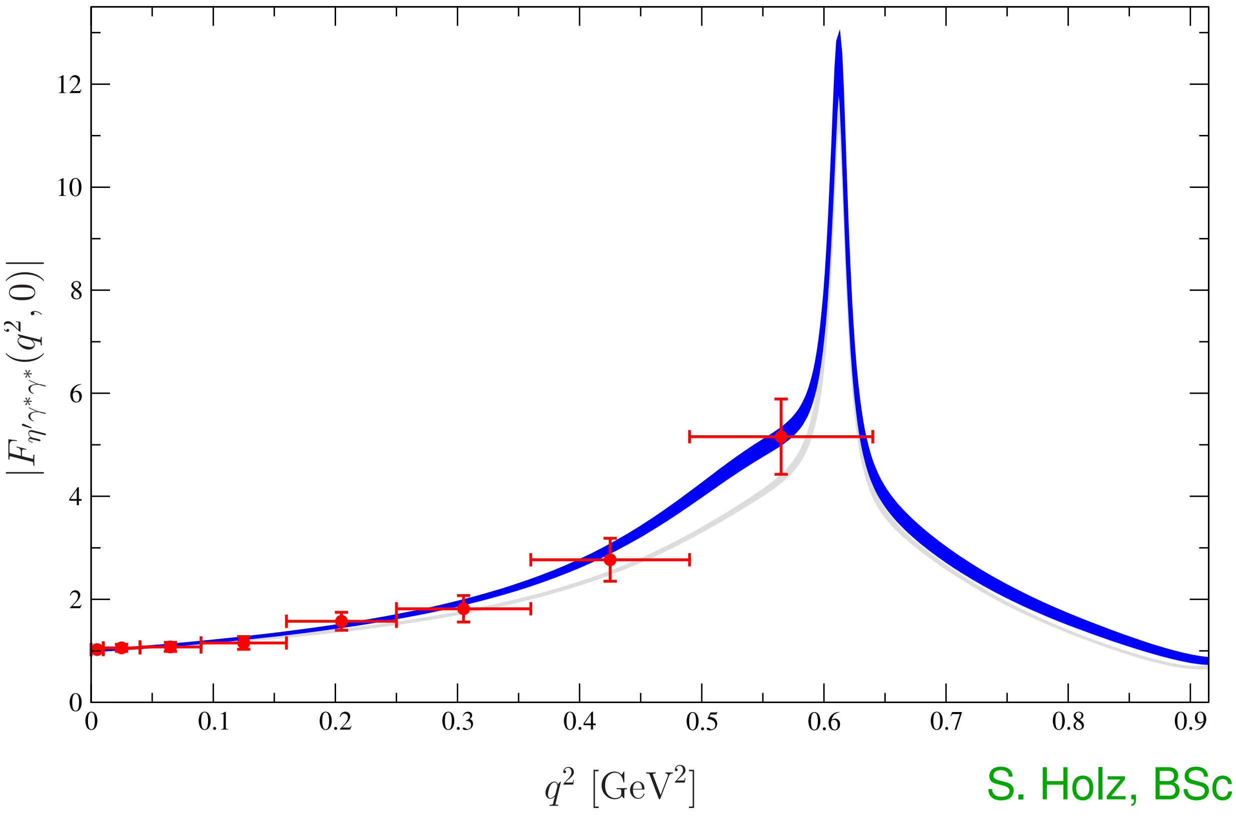
$\longrightarrow$ statistical advantage of hadronic $\eta \to \pi^+\pi^-\gamma$ over direct $\eta \to \ell^+\ell^-\gamma$ (rate suppressed $\propto \alpha^2_{\text{QED}}$)

data: NA60 2009, 2016
A2 2014, 2017

Transition form factor $\eta' \to \gamma^*\gamma$

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$



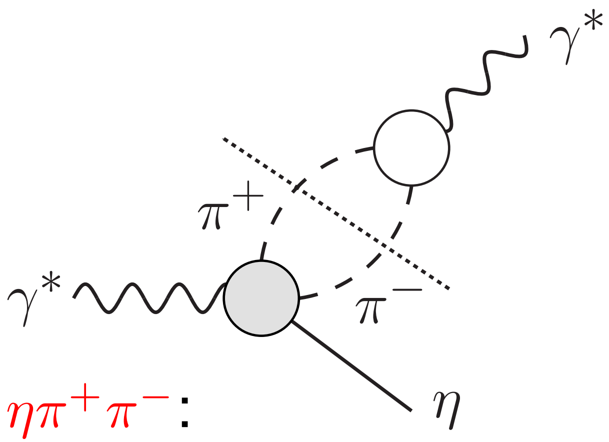


S. Holz, BSc thesis 2016

data: BESIII 2015

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

- idea (again): beat α_{QED}^2 suppression of $e^+e^- \to \eta e^+e^-$ by measuring $e^+e^- \to \eta\pi^+\pi^-$ instead

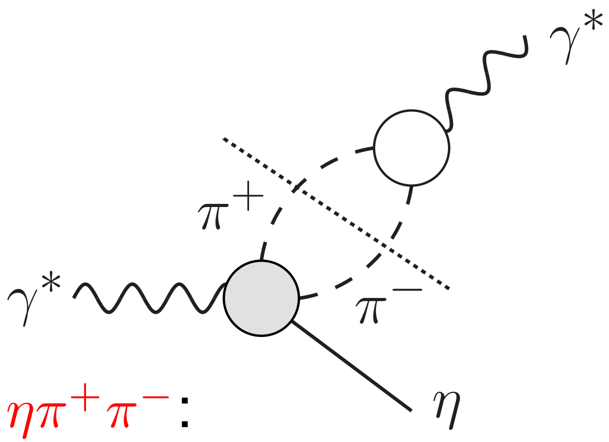


- test factorisation hypothesis in $e^+e^- \to \eta\pi^+\pi^-$:

$$F_{\eta\pi\pi\gamma^*}(t,k^2) \overset{!?}{=} F_{\eta\pi\pi\gamma^*}(t,0) \times \tilde{F}_{\eta\gamma\gamma^*}(k^2)$$

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

- idea (again): beat α^2_{QED} suppression of $e^+e^- \to \eta e^+e^-$ by measuring $e^+e^- \to \eta\pi^+\pi^-$ instead



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▷ allow same form for $F_{\eta\pi\pi\gamma^*}(t,0)$ as in $\eta \to \pi^+\pi^-\gamma$; 3 models:

1. $P^{(1)}(t,0) \times \Omega(t)$, linear function $P^{(1)}(t,0)$
2. $P^{(2)}(t,0) \times \Omega(t)$, quadratic function $P^{(2)}(t,0)$
3. $P^{(a_2)}(t,k^2) \times \Omega(t)$, a_2 left-hand cut
   $\longrightarrow$ induces “natural” factorisation breaking

▷ fit subtractions to $\pi^+\pi^-$ distribution in $e^+e^- \to \eta\pi^+\pi^-$
$\longrightarrow$ are they compatible with the ones in $\eta \to \pi^+\pi^-\gamma$?

Holz, Plenter et al. 2021

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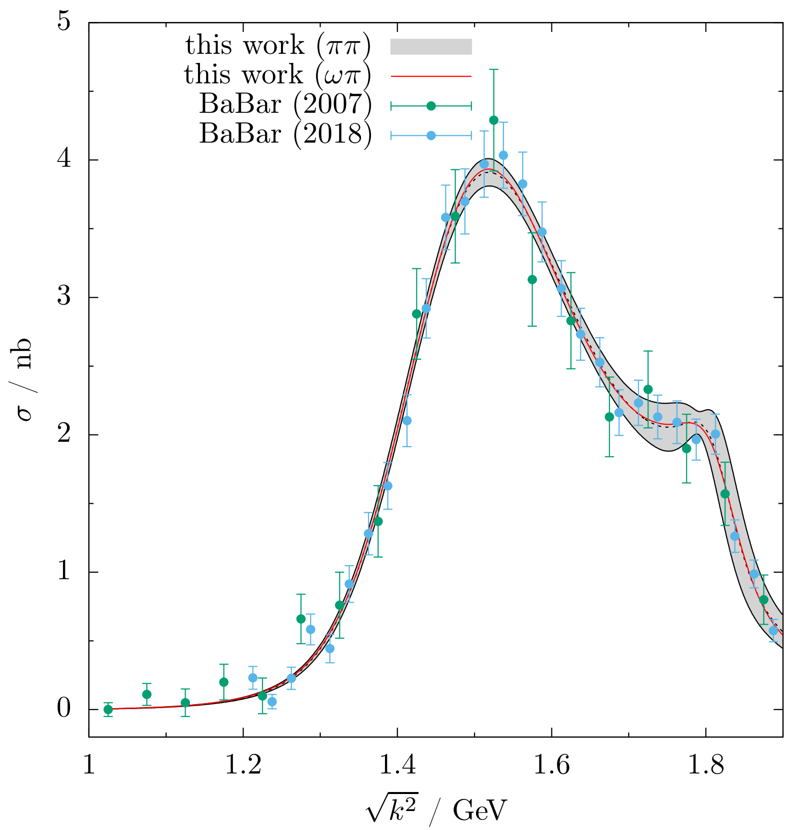
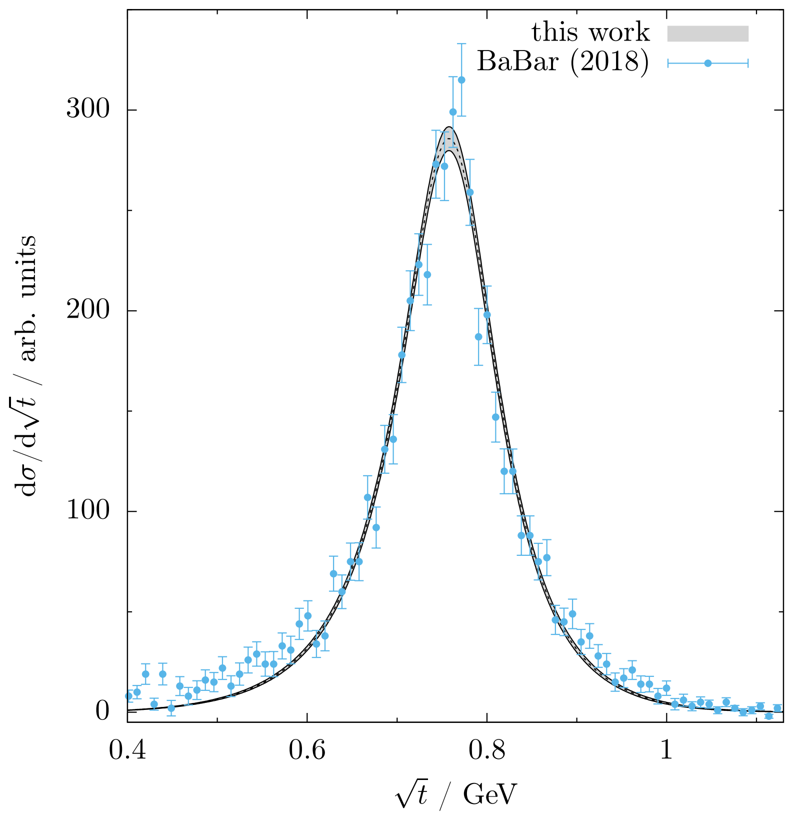
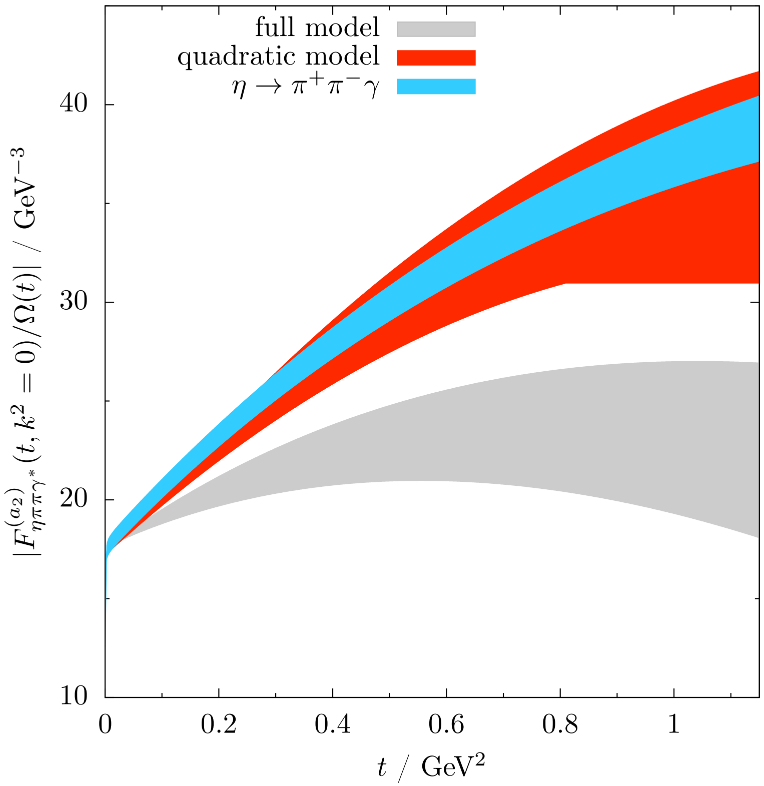


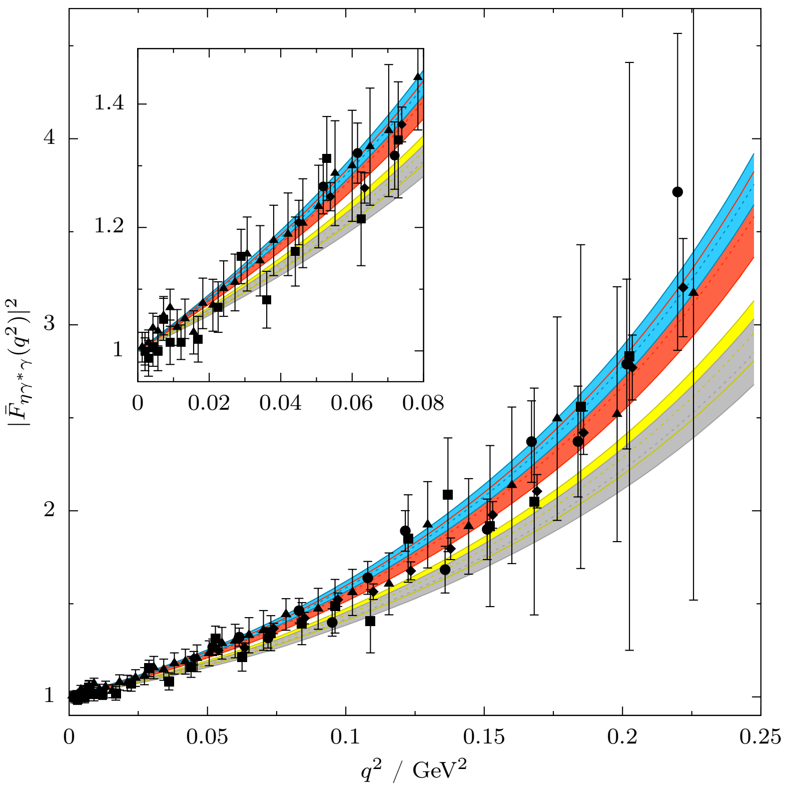


Holz, Plenter et al. 2021; data: BaBar 2007, 2018

- $\tilde{F}_{\eta\gamma\gamma^*}(k^2)$ parameterised by sum of Breit–Wigners (ρ, ρ', ρ'')
- differential spectra $d\sigma/d\sqrt{t}$ integrated over large k^2 range
- $\pi\pi$ spectrum imperfectly described below (?!) the $\rho(770)$ peak

Extrapolation from $e^+e^- \to \eta\pi^+\pi^-$ to $\eta \to \pi^+\pi^-\gamma$

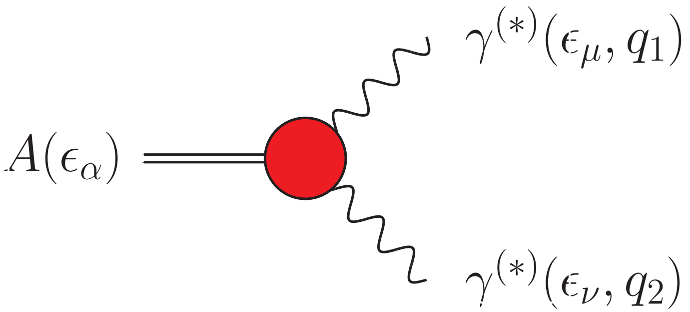




- subtractions fixed from k^2-integrated $\pi\pi$ spectra — compatible with $\eta \to \pi^+\pi^-\gamma$? Holz, Plenter et al. 2021
  - yes with the naïve, factorising, quadratic model
  - no with the physically motivated a_2 model
- extrapolated form factor prediction too low for the full model

Axial-vector transition form factors

- Landau–Yang: no decay into two real photons Landau 1948, Yang 1950
- gauge invariant, singularity-free decomposition into 3 form factors:



$$\mathcal{M}^{\mu\nu\alpha}_{A\gamma^*\gamma^*}(q_1,q_2) = \frac{i}{m_A^2} \sum_{i=a_1,a_2,s} T_i^{\mu\nu\alpha}(q_1,q_2)\mathcal{F}_i(q_1^2,q_2^2)$$

Bardeen, Tung 1968, Tarrach 1975; Hoferichter, Stoffer 2020

- Bose symmetry: under $q_1 \leftrightarrow q_2$
  - ▷ $T_s^{\mu\nu\alpha}(q_1,q_2)$ and $\mathcal{F}_s(q_1^2,q_2^2)$ symmetric
  - ▷ $T_{a_{1/2}}^{\mu\nu\alpha}(q_1,q_2)$ and $\mathcal{F}_{a_{1/2}}(q_1^2,q_2^2)$ antisymmetric

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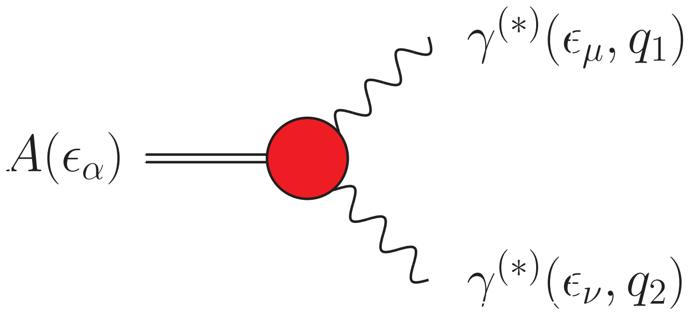

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- here: concentrate on the $f_1(1285) \longrightarrow$ best data basis
- isospin decomposition: isovector–isovector + isoscalar–isoscalar

  SU(3) + mixing information: L3 2007

  isoscalar / isovector $\sim 5\% \longrightarrow$ small correction

Axial-vector transition form factors: asymptotics

- light-cone expansion: $\phi(u) = 6u(1-u)$ Hoferichter, Stoffer 2020

  $\longrightarrow$ talk P. Stoffer; cf. also Leutgeb, Rebhan 2020

$$\mathcal{F}_{a_1}(q_1^2, q_2^2) = \mathcal{O}(q_i^{-6})$$

$$\mathcal{F}_{a_2}(q_1^2, q_2^2) = \underbrace{F_A^{\text{eff}}}_{\text{eff. decay const.}} m_A^3 \int_0^1 du \frac{(2u-1)\phi(u)}{(uq_1^2 + (1-u)q_2^2 - u(1-u)m_A^2)^2} + \mathcal{O}(q_i^{-6})$$

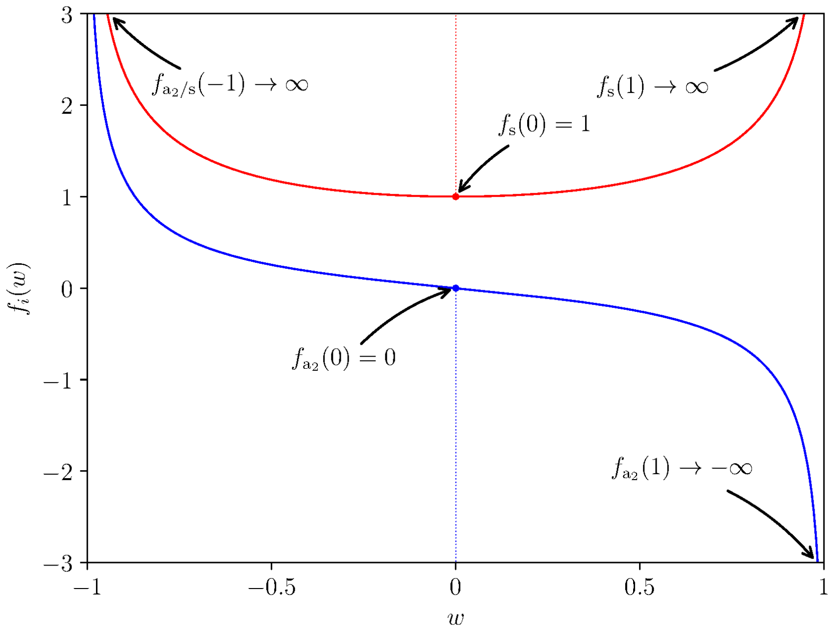
$$\mathcal{F}_s(q_1^2, q_2^2) = -F_A^{\text{eff}} m_A^3 \int_0^1 du \frac{\phi(u)}{(uq_1^2 + (1-u)q_2^2 - u(1-u)m_A^2)^2} + \mathcal{O}(q_i^{-6})$$

- with $Q^2 = \frac{q_1^2+q_2^2}{2}$, $w = \frac{q_1^2-q_2^2}{q_1^2+q_2^2}$:

$$\mathcal{F}_{a_2/s}(q_1^2, q_2^2) = \frac{F_A^{\text{eff}} m_A^3}{Q^4} f_{a_2/s}(w) + \mathcal{O}(Q^{-6})$$

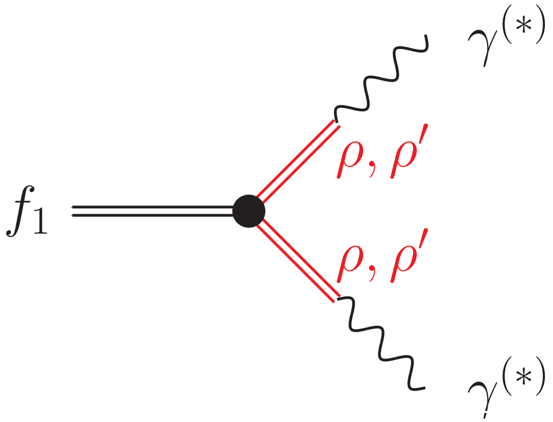
- singly-virt. limits $w = \pm 1$ divergent

  $\longrightarrow$ always suppressed in physical observables/helicity amplitudes



f_1 transition form factors: VMD model

Zanke, Hoferichter, BK 2021



- construct VMD model from $\rho(770)$ and $\rho(1450)$ Breit–Wigners:

$$\mathcal{F}_{a_{1/2}}(q_1^2, q_2^2) = C_{a_{1/2}}\big[(\rho\rho') - (\rho'\rho)\big]$$

- two variants for dominant $\mathcal{F}_s(q_1^2, q_2^2)$:

$$\mathcal{F}_s(q_1^2, q_2^2) = C_s\big[(\rho\rho)\big]$$

$$\tilde{\mathcal{F}}_s(q_1^2, q_2^2) = C_s\Big\{(1-\epsilon_1-\epsilon_2)(\rho\rho) + \frac{\epsilon_1}{2}\big[(\rho\rho') + (\rho'\rho)\big] + \epsilon_2(\rho'\rho')\Big\}$$

$\longrightarrow$ use $\epsilon_{1/2}$ to tune high-energy behaviour:

	$\mathcal{F}_{a_1}(q_1^2, q_2^2)$ $q_{1/2}^2 \approx q^2$	$\mathcal{F}_{a_1}(q_1^2, q_2^2)$ $q_2^2 = 0$	$\mathcal{F}_{a_2}(q_1^2, q_2^2)$ $q_{1/2}^2 \approx q^2$	$\mathcal{F}_s(q_1^2, q_2^2)$ $q_{1/2}^2 = q^2$	$\mathcal{F}_{a_2+s}(q_1^2, q_2^2)$ $q_2^2 = 0$
LCE	$1/q^6$	$1/q_1^6$	$1/q^4$	$1/q^4$	$1/q_1^4$
VMD	$1/q^6$	$1/q_1^2$	$1/q^6$	$1/q^4$	$1/q_1^2$
$\widetilde{\text{VMD}}$	$1/q^6$	$1/q_1^2$	$1/q^6$	$1/q^6$	$1/q_1^4$

$\longrightarrow$ add asymptotic piece above threshold s_m as for π^0

f_1 transition form factors: data input

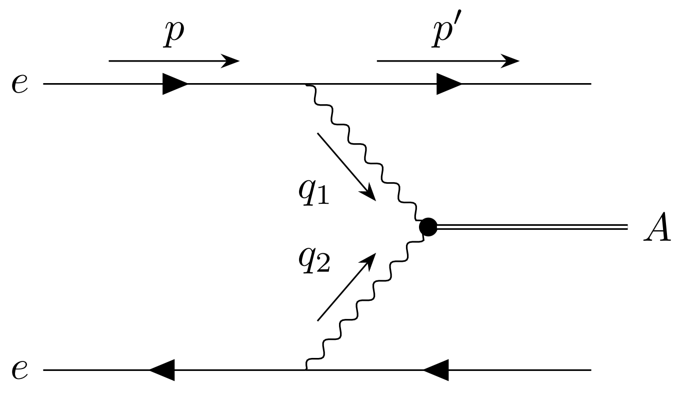
Determination of 3 normalisation constants C_s, C_{a_1}, C_{a_2}

- $e^+e^- \to e^+e^- f_1$: L3 2007

  equivalent two-photon decay width

$$\tilde{\Gamma}_{\gamma\gamma} = \lim_{q_1^2 \to 0} \frac{1}{2} \frac{m_{f_1}^2}{q_1^2} \Gamma(f_1 \to \gamma_L^* \gamma_T)$$

$$= \frac{\pi \alpha_{\text{QED}}^2}{48} m_A |\mathcal{F}_s(0,0)|^2$$



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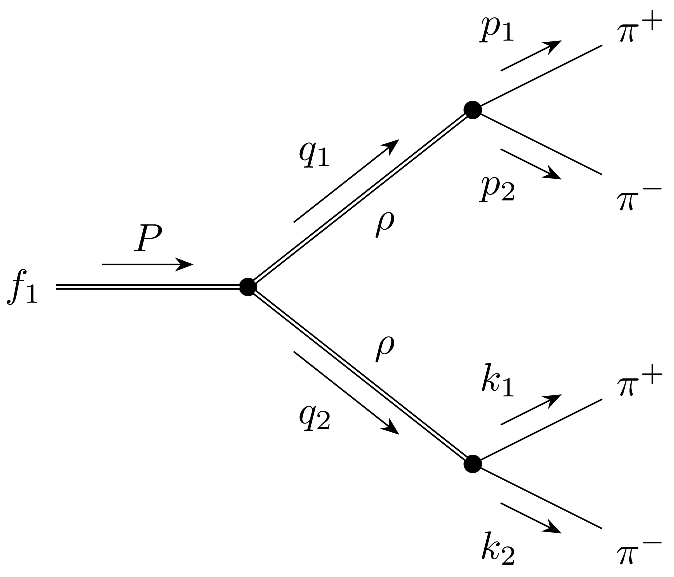

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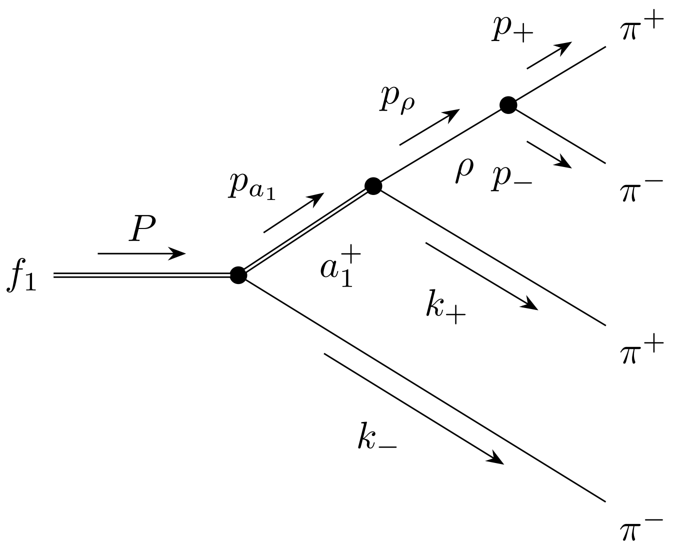

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  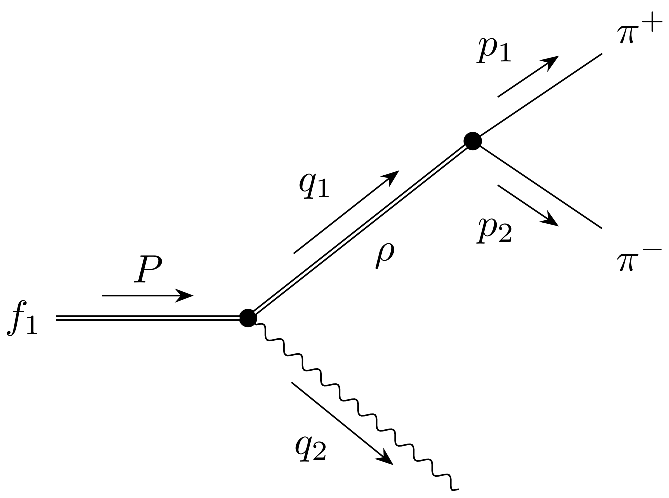

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- $f_1 \to \rho\gamma$: branching ratio + ratio of helicity amps. VES 1995 + ...
  $\longrightarrow$ mainly sensitive to $\mathcal{F}_{a_1}$ and $\mathcal{F}_s$

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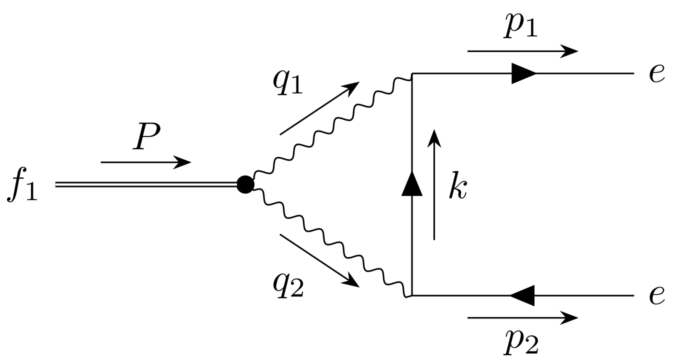

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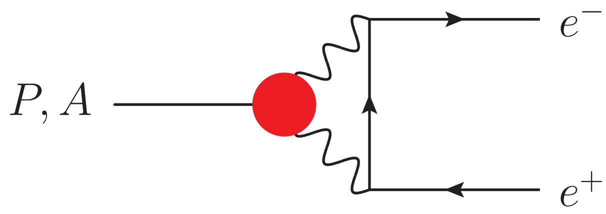
- $\mathcal{B}(f_1 \to e^+e^-) = \left(5.1^{+3.7}_{-2.7}\right) \times 10^{-9}$: SND 2020
  loop effect, sensitive to all 3 form factors cf. also Rudenko 2017

Why the loop-induced e^+e^- decay is interesting



- compare $\pi^0 \to e^+e^-$: TFF $\longrightarrow$ double-spectral function

$$\mathcal{A}_{\pi^0 \to e^+e^-} = \frac{1}{\pi^2} \int_{4M_\pi^2}^{\infty} dx \int_{s_{\text{thr}}}^{\infty} dy \; \rho(x,y) \; K(x,y)$$

$K(x,y)$: kernel $\hat{=}$ loop function with VMD form factor, $x, y \,\hat{=}\, M_V^2$
$\longrightarrow$ can be calculated extremely precisely

Hoferichter, Hoid, BK, Lüdtke 2021

- corresponding expression for $f_1 \to e^+e^-$:

$$\mathcal{A}_{f_1 \to e^+e^-} = D_1 \times C_{a_1} + D_2 \times C_{a_2} + D_3 \times C_s + D_{\text{asym}}$$

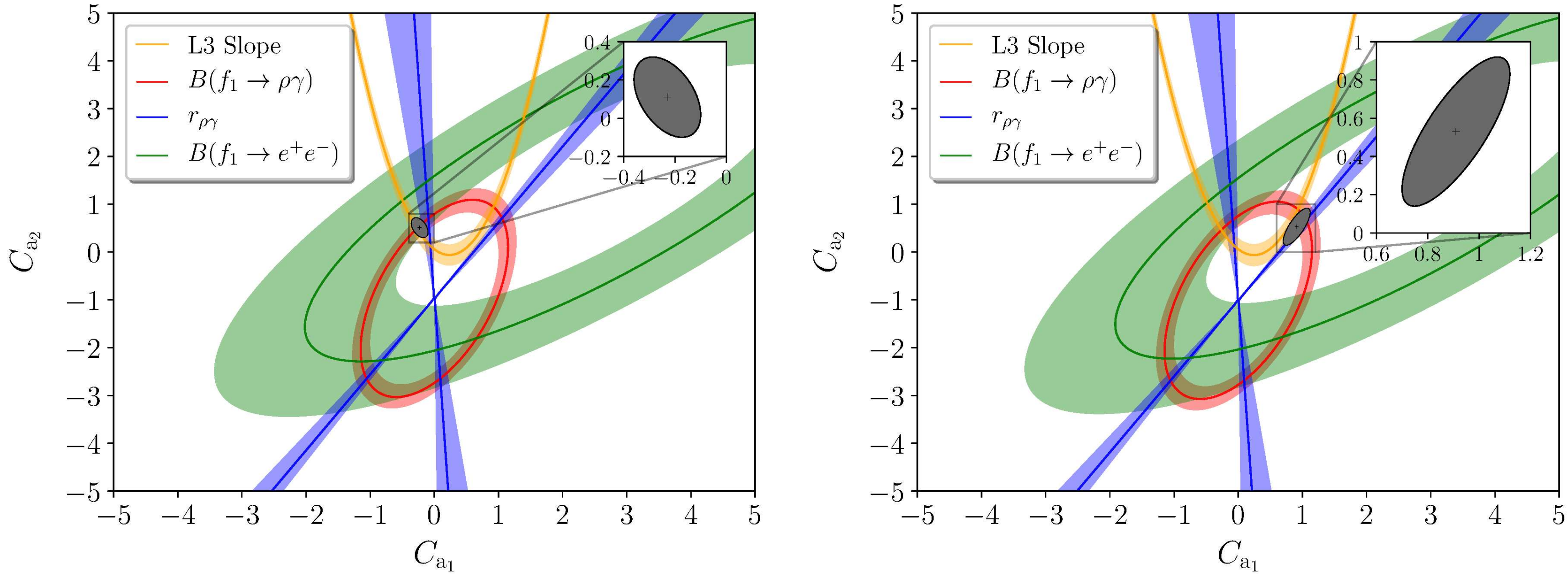
$\longrightarrow$ $D_{1/2/3}$ all same magnitude (D_{asym} small)!

Zanke, Hoferichter, BK 2021

f_1 TFFs: couplings, minimal VMD

Contours in the C_{a_1}–C_{a_2} plane

Zanke, Hoferichter, BK 2021

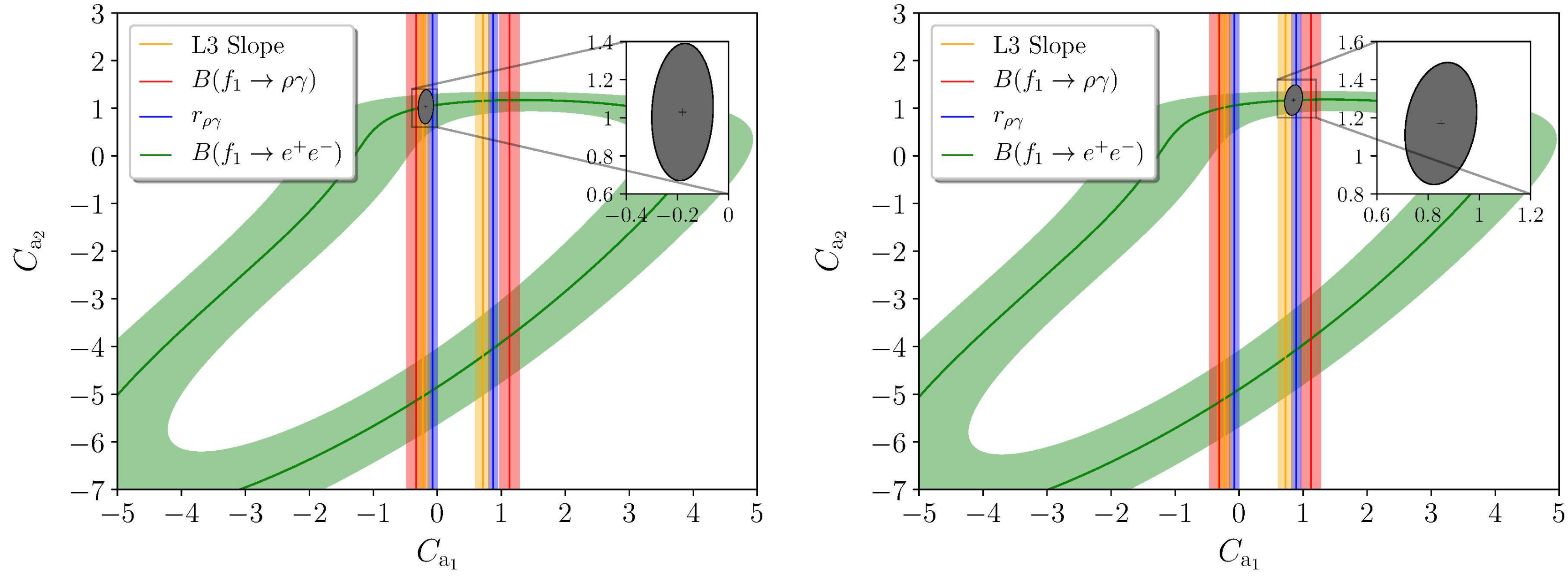


- C_s well determined from $e^+e^- \to e^+e^- f_1$
- always two pairs of solutions for C_{a_1}
- extended VMD: C_{a_2} dependence drops out except in $f_1 \to e^+e^-$

f_1 TFFs: couplings, extended VMD

Contours in the C_{a_1}–C_{a_2} plane

Zanke, Hoferichter, BK 2021



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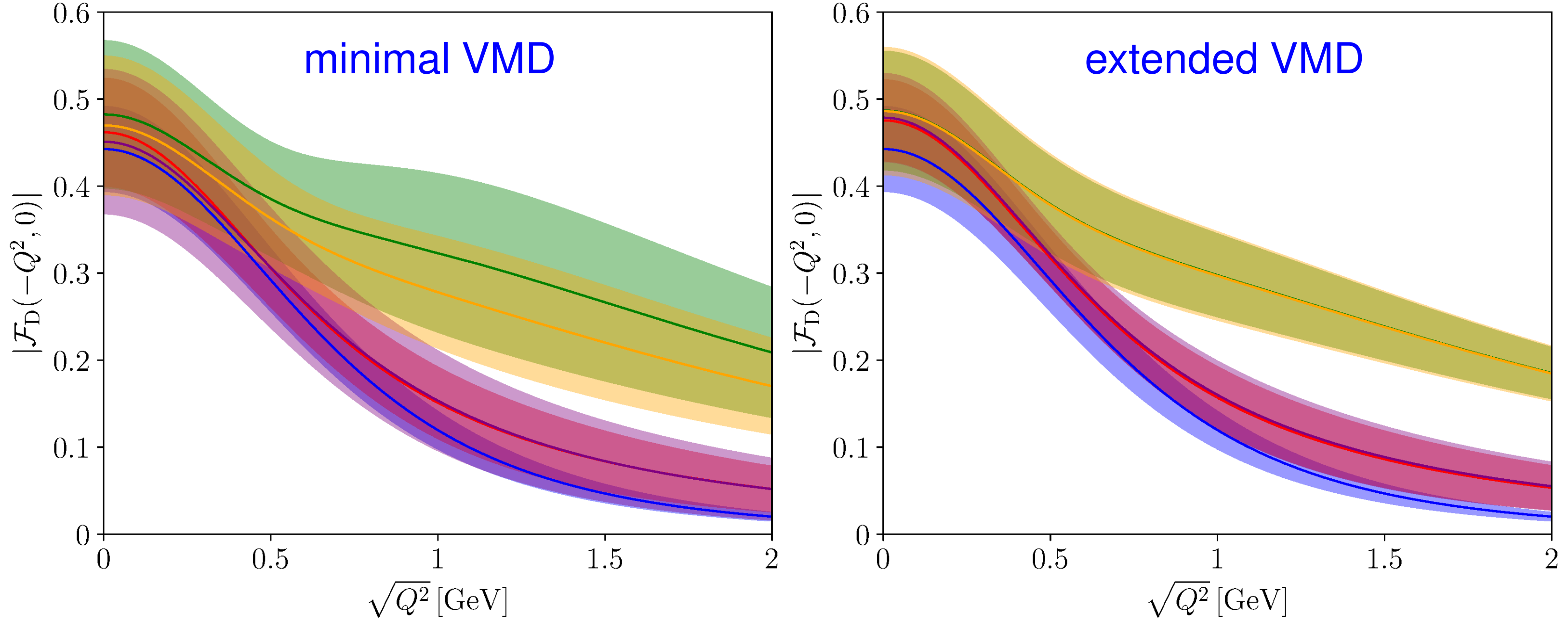
f_1 TFFs: effective form factor

Comparison to effective form factor in $e^+e^- \to e^+e^- f_1$



- Solution 1 agrees well with L3 dipole fit, Solution 2 doesn't
- extended VMD implies asymptotics with

$$F_{f_1}^{\text{eff}}\Big|_{\widetilde{\text{VMD}}} = \frac{C_s M_\rho^2 M_{\rho'}^2}{6m_{f_1}^3} = 95(12)\,\text{MeV} \quad \text{vs.} \quad F_{f_1}^{\text{eff}}\Big|_{\text{L3}} = 86(28)\,\text{MeV}$$

compare to $F_{f1}^{\text{eff}}\Big|_{\text{LCSRs}} = 146(14)\,\text{MeV}$ Yang 2007

Comparison to selected models

- **Quark model**: only $\mathcal{F}_s(q_1^2,q_2^2) = \frac{C_s \times m_A^4}{(m_A^2-q_1^2-q_2^2)^2}$ Schuler et al. 1998
  - ▷ agrees with asymptotic $1/Q^4$, F_A^{eff} too large doubly-virtually
- **Resonance chiral theory**: Roig, Sánchez-Puertas 2020
  - ▷ symmetric TFF vanishes at “leading order”
  - ▷ antisymm. TFFs: no strict VMD, also direct photon coupling
- **Phenomenology**: Rudenko 2017; Milstein, Rudenko 2020
  - ▷ kinematical singularities, complex couplings
- **Factorisation**: $\mathcal{F}_s(q_1^2,q_2^2) = \frac{C_s \times \Lambda_D^4}{(\Lambda_D^2-q_1^2)^2(\Lambda_D^2-q_2^2)^2}$ Pauk, Vanderhaeghen 2014
  - ▷ does not agree with asymptotic constraints
- **Holographic models**: Leutgeb, Rebhan 2020 $\longrightarrow$ talk A. Rebhan
  - ▷ agrees with $1/Q^4$ and w-dependence from Brodsky–Lepage
  - ▷ $F_{f_1}^{\text{eff}}$, C_s reasonable vs. L3, $C_{a_1}=0$, C_{a_2} small
  - ▷ detailed comparison in intermediate range to be done

Summary / Outlook

Towards the doubly-virtual η transition form factor

- high-precision data on $\eta \to \pi^+\pi^-\gamma$ KLOE and $\eta' \to \pi^+\pi^-\gamma$ BESIII allow for high-precision dispersive predictions of $\eta^{(\prime)} \to \gamma\gamma^*$
- $\pi\pi$ spectra in $e^+e^- \to \eta\pi^+\pi^-$ BaBar vs. $\eta \to \pi^+\pi^-\gamma$:
  - ▷ compatible with naïve factorisation
  - ▷ incompatible with dominant left-hand cut

  $\longrightarrow$ hope for better energy-dependent amplitude analysis

Transition form factors for the $f_1(1285)$

- tensor basis & asymptotics clarified
- experimental data insufficient to disentangle 3 TFFs uniquely

  $\longrightarrow$ important role of $e^+e^- \to f_1$ to constrain asymmetric ones